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BIOGRAPHY.

JAMES MATTESON, M. D.

BY F. P. MATZ, M. SC., PH, D.

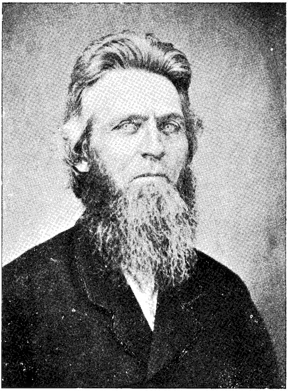
DR. JAMES MATTESON was born at Greenwich, Kent county, R. I., March 28th, 1819. He had few and limited opportunities for improvement in early life, and toiled in factories and on the farm with his father until twenty years of age, when he left the parental roof, relying entirely upon his own energy and industry.

After obtaining a limited education he decided to become a machinist, for which calling he possessed most excellent natural abilities. He accordingly went to Salem, Massachusetts for the purpose of entering upon his chosen pursuit, but was there persuaded to abandon it.

He then turned his attention to school teaching, which he followed for some years in the Eastern States and Pennsylvania. In this worthy and honorable calling he labored earnestly and studied as time and circumstances permitted, and by thus using every available opportunity he obtained a liberal education.

He finally abandoned teaching and emigrated from Pennsylvania to Mt. Upton, Chenango Co., N. Y., where he studied medicine, and graduated at Geneva, N. Y., with honors, in 1849. He then returned to his native state and practiced medicine two years, at the expiration of which time he again returned to the state of New York, and settled in Wayne county, where he was married to Miss Emily Swift, June 18, 1851. After three years' parctice of medicine in that county he located in DeKalb, Illinois, where he remained until his death, December 15, 1876, having died of pulmonary congestion.

He was a well-read physician and a man of much general information.



JAMES MATTESON, M. D.

He possessed extra abilities as a mathematician, and contributed to the *Chronical*, the *Analyst*, and other mathematical papers.

During the last years of his life he often expressed a desire to visit once more his native home in Rhode Island, which desire he gratified after an absence of twenty years in the months of August and September, 1876, visiting many old friends and correspondents and viewing the scenes of his childhood and early youth. He also visited the centennial on his way home.

The life of Dr. James Matteson was not, in the estimation of many a success, because he did not accumulate property and make a show in the world, yet his life was nevertheless a most glorious success. He possessed a great knowledge, and had trained himself to most excellent habits. He was a man of pure moral character and what may truly be called a Christian gentleman, and was trusted and respected in the community where he lived.

That is a successful life in which there is honesty, truth, virtue and Christianity, and these were prominent in the life of Dr. James Matteson. All that he did was done conscientiously, and he would not deviate from what he considered right for any worldly interests which might be offered him.

From 1860 to 1868 he wrote over one hundred mathematical letters to Professor David Trowbridge, besides many to Professors Robinson, Perkins, Evans, and Ficklin, Dr. Hart, and Dr. Hendricks. Some of his problems, of which he originated many, are so difficult that the simplest cases have never been completely solved. In this Department he sometimes appeared over the pseudonym of "Iago." A few years ago he published a pamphlet containing a hundred and twenty-five choice problems. In 1888, Dr. Artimas Martin published a collection of 24 Diophantine Problems and their Solutions, which were compiled by Dr. Matteson. One of his problems is styled "The Elephant," it being so ponderous as to include a score or two of problems in Diophantine Analysis, of which branch of mathematics, and also with Indeterminate Analysis, he was especially fond.

Dr. Matteson was a Baptist by profession, and one in whom religion and high moral worth were paramount to mere *denominational servitude.* A few years ago he with a few others succeeded in *bringing a preacher to justice,* (a task always very difficult of performance) for doing which he suddenly found his connection with the church had been severed.

BIOGRAPHY.

SAMUEL GARDNER CAGWIN.

BY B. F. BURLESON.

SAMUEL GARDNER CAGWIN was born in Oneida county, State of New York about sixty years ago. He had only a cammon school education with the exception of a term or two in the New London Academy.

He had an inordinate love for mathematics, and I have often heard him tell how he astonished his teachers by solving problems to which they could in no way make any attempt. His great predilection was for Astronomy, in which he took great delight in poring over half of the night. I know I had a copy of Loomis' Astronomy which I loaned him, and which he kept till he died. It has come back to me since his death. This work has about fifty difficult examples in the back part of the book. It has returned to me with all the answers carefully set down on one of the blank leaves of the book, the work of his hands.

I have seen him go out and carefully measure his shadow cast on a level board, and come in and with the length and the time of the year, and the latitude, tell the time of day the shadow was measured.

I have staid many nights with him as I taught school one year in the old Academy that he attended in New London. I know of his insatiable love for mathematics. Almost every night he would tire me out, and I would go to bed and leave him poring over his problems while he would remain up till after midnight: and yet he would be up at five o'clock in the morning calling to his boys to help milk, for he kept a large dairy. Working in this way his iron constitution had to succomb at last. He got up one morning about two years ago and undertook to build a fire in his stove. He was trying to do it a long time. Finally he took out the ashes to dump them, but he forgot what he was about and brought them back. His wife saw him and spoke to him. He could not answer. The first stroke of paralysis had set its deadly mark upon him. Other shocks followed, and then his mind gave away. They took him to the Hospital at Utica, where the writer of this article was at the time, laboring under pretty much the same difficulty. I was nearly well when he came, and I saw at once he would not live long. He could not talk intelligently on any subject. He kept getting worse and died in the last days of June, a few months after I was discharged.

In disposition, Mr. Cagwin was kind and friendly. He was a very hard working man. If he spent some time at his beloved mathematics he made up for the stolen time by doing two hours' work in one. I heard him tell how in the hay field a person once gave him a problem. It seems that a man was hired for one cent for his first day's work, two cents for his second, four for his third, etc., doubling each previous day's earning. If hired for a year what would be his entire earnings? There was a dispute about his wages. He got off from a load of hay, and went to the house, and solved it in order to find out what it would be. You may be sure the hay flew after that fast enough.

Such was the man. In him was the making of a Galileo, perhaps, but like Burns, he was tied to earth at the tail end of a plow.

"Poor fool! the base and soulless wardling cries
To spend his time for naught
Why did he not to pleasure give his days
His nights to rest, and live while live he might?"

For a reply to this startling inquiry we refer the reader to the whole poem to which these lines are the introduction.

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

CHAPTER III.

In determining the equations of lowest form upon which depends the inscription of a regular polygon, it might appear that we can employ a random grouping of the chords. But this grouping is *not* arbitrary but according to a profound principle of the Theory of Numbers, the exposition of which is beyond the scope of this paper.

For example, the six chords of the regular 13-gon were formed into two groups, $(A_1+A_3-A_4)$ and $(A_5-A_2-A_6)$. By any different grouping, the product of the two groups could not be expressed rationally in terms of the unit radius, but would involve the chords themselves.

The ordinary rule may be stated thus: Let n be the number of sides of the regular polygon and write $p=\frac{n-1}{2}$. The subscripts are written in the definite order: 1, its double 2, its double 4, etc, until a number $>p$ is obtained when we take the difference between it and n, take its double, etc, at last obtaining the number p.

Thus for $n=13$, we get the following *order* of the subscripts: 1,2,4,5,3,6.

To group them into *two* groups, choose every second subscript, thus: 1, 4, 3 and 2, 5, 6.

To group them into *three* groups, choose everp third subscript, thus: 1, 5; also 2, 3; also 4, 6.

In some rare cases (as that of the regular 257-gon) we must treble the subscript each time instead of doubling it to obtain the required *order*.

Let us consider the regular polygon of 17 *side.*
It was with rare foresight that the youthful discoverer Gauss declared to his room-mate that the discovery of its geometric inscriptibility was alone sufficient to make his name immortal.

The sum of the two groups $(A_1-A_2-A_4-A_8)$ and $(A_3+A_5-A_6+A_7)$ equals 1. Their product expanded and reduced equals $-4(A_1-A_2+A_3-A_4+A_5-A_6+A_7-A_8)=-4$. Hence, the two groups are the roots of the quadratic $x^2-x-4=0$.

Hence, $A_1-A_2-A_4-A_8=\frac{1}{2}(1-\sqrt{17})$: $A_3+A_5-A_6+A_7=\frac{1}{2}(1+\sqrt{17})$.

Forming the sub-groups, (A_1-A_4) and $(-A_2-A_8)$, their product $=-(A_1-A_2+A_3-A_4+A_5-A_6+A_7-A_8)=-1$.

Similarily, $(A_3+A_5)(-A_6+A_7)=-1$.

Hence, (A_1-A_4) and $(-A_2-A_8)$ are the roots of $x^2-\frac{1}{2}(1-\sqrt{17})x-1=0$. Also (A_3+A_5) and $(-A_6+A_7)$ are the roots of $x^2-\frac{1}{2}(1+\sqrt{17})x-1=0$.

$\therefore A_1-A_4=\frac{1}{4}[1-\sqrt{17}+\sqrt{2(17-\sqrt{17})}]$;$-A_2-A_8=\frac{1}{4}[1-\sqrt{17}-\sqrt{2(17-\sqrt{17})}]$

$A_3+A_5=\frac{1}{4}[1+\sqrt{17}+\sqrt{2(17+\sqrt{17})}]$; $-A_6+A_7=\frac{1}{4}[1+\sqrt{17}-\sqrt{2(17+\sqrt{17})}]$

Now $A_1.A_4=A_3+A_5$; $A_2.A_8=A_6-A_7$

$A_3.A_5=A_2+A_8$; $A_6.A_7=A_1-A_4$.

Hence, A_1 and $-A_4$ are the roots of the quadratic

$$x^2-\tfrac{1}{4}[1-\sqrt{17}+\sqrt{2(17-\sqrt{17})}]x-\tfrac{1}{4}[1+\sqrt{17}+\sqrt{2(17+\sqrt{17})}]=0.$$

Thus $A_1=\frac{1}{8}[1-\sqrt{17}+\sqrt{[2(17-\sqrt{17})]}$

$$+\sqrt{68+12\sqrt{17}+16\sqrt{[2(17+\sqrt{17})]}+2(1-\sqrt{17})\sqrt{[2(17-\sqrt{17})]}}].$$

Similarly for the other seven chords.

In a later chapter will be given a beautiful Geometric construction for inscribing the regular 17-gon founded upon the method invented by Von Staudt for graphically finding the roots of any quadratic.

For the regular 19-gon, the order of the subscripts is 1,2,4,8,3,6,7,5,9. Hence the groups are $(A_1-A_8+A_7)$,$(-A_2+A_3+A_5)$, and $(-A_4-A_6+A_9)$.

$(A_1-A_8+A_7)(-A_2+A_3+A_5)$, expanded gives $(-A_1-2A_2+2A_3-3A_4+2A_5-3A_6+A_7-A_8+3A_9)=-[1+(-A_2+A_3+A_5)+2(-A_4-A_6+A_9)]$.

Similarly, $(A_1-A_8+A_7)(-A_4-A_6+A_9)=-[1+(A_1-A_8+A_7)+2(-A_2+A_3+A_5)]$ $(-A_2+A_3+A_5)(-A_4-A_6+A_9)$

$$=-[1+(-A_4-A_6+A_9)+2(A_1-A_8+A_7)].$$

Write $A=(A_1-A_8+A_7)$; $B=(-A_2+A_3+A_5)$; $C=(-A_4-A_6+A_9)$ $\therefore$ $A+B+C=1$. $AB=-(1+B+2C)$; $AC=-(1+A+2b)$; $BC=-(1+C+2A)$. $\therefore$ $AB+AC+BC=-[3+3(A+B+C)]=-6$. $ABC=-A(1+C+2A)=-A(3-2B-C)=-(3A-2AB-AC)=-(3A+2+2B+4C+1+A+2B)=-[3+4(A+B+C)]=-7$. Hence, A, B, C are the three roots of the cubic $x^3-x^2-6x+7=0$. Now $(A_1.A_7-A_1.A_8-A_7.A_8)=-[(A_1-A_8+A_7)+(-A_4-A_6+A_9)]=-(A+C)=B-1$. $A_1.A_7.A_8=A_1(A_1-A_4)=2+(A_2-A_3-A_5)=2-B$. Hence, $A_1,-A_8$, and A_7 are the roots of $x^3-Ax^2+(B-1)x+(2-B)=0$. Similarly, $-A_2,A_3$, and A_5 are the roots of $x^3-Bx^2+(C-1)x+(2--C)=0$. Similarly, $-A_4,-A_6$, and A_9 are the roots of $x^3-Cx^2-(A-1)x+(2-A)=0$.

For the regular 31-gon, we find that $(A_1-A_2-A_4-A_8+A_{15})$, $(A_3-A_6+A_7-A_{12}-A_{14})$, and $(A_5+A_9-A_{10}+A_{11}+A_{13})$ are the three roots, say A, B, C, of $x^3-x^2-10x+8=0$. Further, that $A_1,-A_2,-A_4,-A_8,+A_{15}$ are the five roots of $x^5-Ax^4-(1+B)x^3-(1+2B+3C)x^2-2(1+C)x+1=0$.

That $A_3,-A_6,+A_7,-A_{12},-A_{14}$ are the roots of $x^5-Bx^4-(1+C)x^3-(1+2C+3A)x^2-2(1+A)x+1=0$. That $A_5,A_9,-A_{10},A_{11},A_{13}$ are the roots of $x^5-Cx^4-(1+A)x^3-(1+2A+3B)x^2-2(1+B)x+1=0$.

The above examples will serve to indicate the spirit of my method. In a lengthy memoir I have given a general and exhaustive treatment of the subject based on these and similar geometric principles and upon the Theory of of Numbers. It is proved that, for a regular polygon of n sides, n being a prime number, to be *geometrically* inscriptible, n must be of the form 2^m+1. For example, those of 3, 5, 17, 257, 65537, etc, sides.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

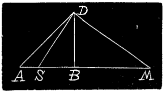
By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph.D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the October Number.]

PROPOSITION X.

If the straight DB stands at right angles (fig 8.) to a certain ABM, and the join DM be greater than the join DA, also the base B,M will be greater than the base BA.

And inversely.



PROOF. And in the first place assuredly these bases will not be mutually equal. Otherwise (Eu. I. 4.) AD and DM would be equal, contrary to the hypothesis. But neither will BA be greater than BM. Otherwise, in BA the portion BS being taken equal to BM, and SD joined, the angles BSD, BMD (Eu. I. 4.) would be equal: But the angle BSD is (Eu. I. 16.) greater than the angle BAD. Therefore the angle BMD would be greater than this.

But this is contrary to Eu. I. 18; since the side DM in the triangle MDA is supposed greater than the side DA.

It remains therefore, that the base BM is greater than the base BA.

Quod erat primo loco demonstrandum.

Next if either base, as BA suppose (the figure need not be changed) is conceived as greater than the other BM; then the join DS, which cuts off from BA the portion SB equal to BM, will be equal (Eu. I. 4.) to the join DM.

Again the angle DSA will be obtuse (Eu. I. 16.) and the angle DAS acute (Eu. I. 17.)

Wherefore (Eu. I. 18.) the join DA will be greater than the join DS, and the join supposed equal to it DM. Quod erat secundo loco demonstrandum.

Itaque constant proposita.

PROPOSITION XI.

Let the straight AP (as long as you choose) cut the two straights PL, AD (fig.9.) the first indeed at right angles in P, but the latter at A in any acute angle converging toword PL. I say the straights AD, PL (in the hypothesis of right angle) will at length, meet in some point, and indeed at a finite or terminated distance, if they are prolonged toward that side on which they make with the transversal AP two angles together less than two right angles.

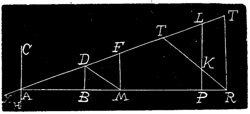
PROOF. Prolong DA toward the other side to some point X, and through A erect to AP the perpendicular HAC, the point H being on the side of the angle XAP.

Then in AD produced toward the side of PL assume two equal inter-

vals AD, DF, and let fall upon AP the perpendiculars DB, FM, which assuredly fall (Eu. I. 17.) on the side of the acute angle DAP, and join DM.

I ought to show that the join DM will be equal to DF, or DA.

And in the first place indeed DM cannot be greater than DF. For otherwise the angle DMF would be less (Eu. I. 18.) than the angle DFM, or its equal (P. VIII, in the hypothesis of right angle) the angle XAH, or its vertical CAD.



Wherefore (since the angles CAM, FMA are assumed equal, as being right) the remaining angle DMA would be greater than the remaining angle DAM. But this is absurd (against Eu. I. 18.) if indeed DM be taken greater than DF, or DA.

But neither will DM be less than this DF. Otherwise the angle DMF would be greater (Eu. I. 18.) than the angle DFM, or its equal (P. VIII, in the hypothesis of right angle) the angle XAH, or its vertical CAD. Wherefore again, as above, the remaining angle DMA will not be greater, but less than the remaining angle DAM. But this is absurd (against Eu. I. 18.) since assuredly DM is taken less than DF, or DA.

It remains therefore, that the join DM is equal to DF, or DA. Wherefore in the triangle DAM (Eu. I. 5.) the angles at the points A, and M will be equal; and therefore in the triangles DBA, DBM, right-angled at B, the bases AB, BM will be equal (Eu. I. 26).

This indeed was here our aim.

Since therefore, (assuming in AD produced the interval AF double the interval AD) the perpendicular FM dropped on the transversal AP cuts off from AP toward P a base AM double AB, which the perpendicular let fall from the point D cuts off; it is manifest that this duplication of the preceding interval can be so many times repeated, that thus in AD continued we attain to a certain point T, from which the perpendicular let fall upon AP prolonged cuts off a certain AR greater than the finite AP however great.

But it holds, that this cannot happen, unless after the meeting of this prolonged AD with PL in some point L.

For if the point T occured before this meeting, the perpendicular TR must cut PL in some point K. But then in the triangle KPR would be found two right angles at the points P, and R; which is absurd (against Eu. I. 17.).

Therefore it holds that the straights AD, PL meet each other mutually (in the hypothesis of right angle) in some point (and indeed at a finite, or terminated distance) if they be produced toward that side, on which with the transversal AP (of finite length however great) they make two angles together less than two right angles. Quod erat demonstrandum.

NOTE ON AREAS AND VOLUMES.

By G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

There is so much interest centred around the curve $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the surface $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1$, that it is necessary to give to the mathematicians a general formula for each that will hold for any positive integral values of m, n, p.

$$A=\text{ area }=4\int\int dxdy=\frac{4\,ab}{\dfrac{4}{(2m+1)(2n=1)}}\cdot\frac{\Gamma\left(\dfrac{2m+1}{2}\right)\Gamma\left(\dfrac{2n+1}{2}\right)}{\Gamma\left(\dfrac{2m+1}{2}+\dfrac{2n+1}{2}+1\right)}$$

$$=\frac{4ab}{\dfrac{2}{2m+1}+\dfrac{2}{2n+1}}\cdot\frac{\Gamma\left(\dfrac{2m+1}{2}\right)\Gamma\left(\dfrac{2n+1}{2}\right)}{\Gamma\left(\dfrac{2m+1}{2}+\dfrac{2n+1}{2}\right)},$$ by Dirichlet's theorem under the above conditions.

$$\therefore\ A=\frac{ab(2m+1)(2n+1)}{(m+n)(m+n+1)}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(m+n)}\ \ldots.(1)$$

$$=\frac{1.3.5\ldots.(2m+1)\times 1.3.5\ldots.(2n+1)}{2.4.6\ldots.2(m+n+1)}\cdot 2\pi ab.$$

When $m=n=0$, $A=\pi ab$, area of the curve $\left(\frac{x}{a}\right)^{2}+\left(\frac{y}{b}\right)^{2}=1$, the ellipse.

When $m=n=1$, $A=\frac{3}{8}\pi ab$, area of the hypocycloid, $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

When $m=0$, $n=1$, $A=\frac{3}{4}\pi ab$, area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

When $m=n=2$, $A=\frac{15}{128}\pi ab$, area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1$.

When $m=1$, $n=2$, $A=\frac{15}{64}\pi ab$, area of curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1$.

When $m=n=4$, $A=\dfrac{1.5.7.9.\pi ab}{(32)^3}$, area of the curve $\left(\frac{x}{a}\right)^{\frac{2}{9}}+\left(\frac{y}{b}\right)^{\frac{2}{9}}=1$.

$V=\text{volume}=8\int\int\int dxdydz$

$$= \frac{\dfrac{8abc}{8}}{(2m+1)(2n+1)(2p+1)} \cdot \frac{\Gamma\left(\dfrac{2m+1}{2}\right)\Gamma\left(\dfrac{2n+1}{2}\right)\Gamma\left(\dfrac{2p+1}{2}\right)}{\Gamma\left(\dfrac{2m+1}{2}+\dfrac{2n+1}{2}+\dfrac{2p+1}{2}+1\right)}$$

$$= \frac{8abc}{\dfrac{4}{(2m+1)(2n+1)}+\dfrac{4}{(2n+1)(2p+1)}+\dfrac{4}{(2m+1)(2p+1)}} \times$$

$$\frac{\Gamma\left(\dfrac{2m+1}{2}\right)\Gamma\left(\dfrac{2n+1}{2}\right)\Gamma\left(\dfrac{2p+1}{2}\right)}{\Gamma\left(\dfrac{2m+1}{2}+\dfrac{2n+1}{2}+\dfrac{2p+1}{2}\right)}$$

$$= \frac{4abc(2m+1)(2n+1)(2p+1)}{(2m+2n+2p+3)(2m+2n+2p+1)} \cdot \frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{\Gamma(m+n+p+\frac{1}{2})} \ldots\ldots (2)$$

$$= \frac{1.3.5\ldots\ldots(2m+1)\times 1.3.5\ldots\ldots(2n+1)\times 1.3.5\ldots\ldots(2p+1)}{1.3.5\ldots\ldots(2m+2n+2p+3)} \cdot 4\pi abc.$$

When $m=n=p=0$, $V=\frac{4}{3}\pi abc$, volume of $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1.$

When $m=n=p=1$, $V=\frac{4}{35}\pi abc$, volume of $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1.$

When $m=n=p=2$, $V=\frac{4.\ 5\ \pi abc}{3.7.11.13}$, volume of $\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}+\left(\frac{z}{c}\right)^{\frac{2}{5}}=1.$

When $m=n=p=3$, $V=\frac{4.5.7\pi abc}{9\ 11.13.17.19}$, volume of $\left(\frac{x}{a}\right)^{\frac{2}{7}}+\left(\frac{y}{b}\right)^{\frac{2}{7}}+\left(\frac{z}{c}\right)^{\frac{2}{7}}=1$

When $m=0$, $n=1$, $p=2$, $V=\frac{4}{21}\pi abc$, volume of $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{5}}=1.$

When $m=1$, $n=2$, $p=3$, $V=\frac{4\pi abc}{3.11.13}$, volume of $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}+\left(\frac{z}{c}\right)^{\frac{2}{7}}=1.$

Formulae (1), and (2) will do for any admissible values of m, n, p.

Let $m=n=p=\frac{3}{2}$; then $V=\frac{4abc\times 4.4.4}{12.\ 10} \cdot \frac{[\Gamma(2)]^3}{\Gamma(5)} = \frac{4}{45}\,abc,$

the volume of $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1.$

When $m=n=\frac{3}{2}$, $A=ab\frac{4.\ 4}{4.\ 3} \cdot \frac{[\Gamma(2)]^2}{\Gamma(3)} = \frac{2}{3}ab$, the area of the

curve $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1.$

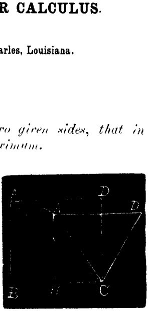
The above formulae have been expressed in a little different form in the *Mathematical Magazien*, but they are so useful that they will bear repetion here.

ISOPERIMETRY WITHOUT CURVES OR CALCULUS.

By Professor P. H. PHILBRICK, M. Sc., C. E., Lake Charles, Louisiana.

[Continued from the October Number.]

PROPOSITION III. *Of all triangles formed with two given sides, that in which these sides are perpendicular to each other is a maximum.*



Let ABC and $A'BC$ be two triangles having $AB = A'B$, BC common, and ABC a right-angle.

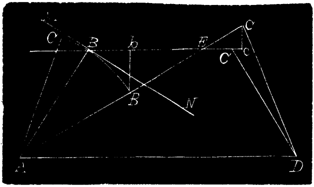
Now $A'H < A'B = AB$. Hence the bases being the same, and the altitude of the one less than the altitude of the other, the area is also less.

PROPOSITION IV. *Any rectangle is greater than any rhomboid, if their bases and perimeters are equal.*

Completing the parallelograms $ABCD$ and $A'BCD'$, we have, $ABCD > A'BCD'$, since they are respectively double the triangles ABC and $A'BC$.

COROLLARY. It follows that any square is greater than any rhombus of equal perimeter.

PROPOSITION V. *Of all quadrilaterals having the same sides, three being equal; that which has equal angles between equal sides, has the maximum area.*



Let $ABCD$ be such a quadrilateral in which $AB = BC = CD$, and $ABC = BCD$. And let us suppose those sides to take the position AB', BC' and $C'D$, the side AD remaining fixed.

Draw MBN perpendicular to AB. Drop the perpendiculars $B'b$ and $C'c$ on BC and BC prolonged, and prolong CB to the left. Make the triangle $ABC'' =$ the triangle DCC'. Now angle $B'Bb > NBb$ and angle $C'Cc = C''BK < MBK = NBb$. $\therefore$ $C'Cc < B'Bb$. Now $bE < B'E$ and $cE < C'E$: adding gives $bc < B'C'$ or $bC + Cc < B'C' = BC$.

But $Bb + bc = BC$. $\therefore$ $Bb + bC > bC + Cc$ or $Bb > Cc$. And (since $C'Cc < B'Bb$), still more is $C'c < Bb$, and $CC' < BB'$. Hence, triangle $DCC' <$ triangle ABB'. Moreover, since $cC' < bB'$; $C'E < B'E$, $cE < bE$, and still more is $CE < BE$.

Hence triangle $ECC' <$ triangle EBB'. But, $ABCD = AB'ECD + ABB' + EBB'$ and $AB'C'D = AB'ECD + DCC' + ECC'$.

Hence $ABCD$ is greater than $AB'C'D$.

COROLLARY. It follows, that the quadrilateral of three equal sides, and maximum area, is a trapezoid; that the angles including the fourth side are also equal; that the opposite angles are supplementary; and that the trapezoid is inscriptible.

[*To be continued.*]

PROFESSOR SYLVESTER'S RECIPROCANTS.

By F. P. MATZ, M. Sc., Ph. D., New Windsor, Maryland.

To those functions of the successive derivatives of y with respect to x, which preserve their form unaltered, except for dy/dx as a factor, when the independent and dependent variables x and y are interchanged, Professor Sylvester gave the name of *Reciprocants*.

According to the general theory with respect to the inversion of the indepent and dependent variable, we must have the relations:

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy};\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(1 \Big/ \frac{dx}{dy}\right) = \frac{d}{dy}\left(1 \Big/ \frac{dx}{dy}\right)\frac{dy}{dx} = -\frac{d^2x}{dy^2} \Big/ \left(\frac{dx}{dy}\right)^3;$$

$$\frac{d^3y}{dx^3} = \frac{d}{dy}\left[-\frac{d^2x}{dy^2} \Big/ \left(\frac{dx}{dy}\right)^3\right]\frac{dy}{dx} = -\left[\frac{dx}{dy}\cdot\frac{d^3x}{dy^3} - 3\left(\frac{d^2x}{dy^2}\right)^2\right] \Big/ \left(\frac{dx}{dy}\right)^5;$$

$$\frac{d^4y}{dx^4} = -\left[\left(\frac{dx}{dy}\right)^2\left(\frac{d^4x}{dy^4}\right) - 10\,\frac{dx}{dy}\cdot\frac{d^2x}{dy^2}\cdot\frac{d^3x}{dy^3} + 15\left(\frac{d^2x}{dy^2}\right)^3\right] \Big/ \left(\frac{dx}{dy}\right)^7;\ \text{etc.}$$

After these relations are substituted for the various differential coefficients of y with respect to x, in any function of these differential coefficients or derivatives, we are said to have interchanged the independent and dependent variable.

Assume $dy/dx = T$, $d^2y/dx^2 = A\,\underline{|2}$, $d^3y/dx^3 = B\,\underline{|3}$, $d^4y/dx^4 = C\,\underline{|4}$, etc.; then, after eliminating the constants in the general equation of the straight line, by the method of differentiation, we obtain $d^2y/dx^2 = A, = d^2x/dy^2 = 0$ ....(1).

The left-hand member of (1) is Professor Sylvester's first *pure* reciprocant, since it does not involve dy/dx; and this reciprocant is briefly and typically expressed by A. The third member of (1) represents the reciprocant when the independent and dependent variables x and y are interchanged.

The equation of the parobola $(\alpha x+\beta y)^2+2gx+2fy+c=0$, in which $\alpha^2=a$ and $\beta^2=b$, gives the second pure reciprocant:

$$3\frac{d^2y}{dx^2}\cdot\frac{d^4y}{dx^4}-5\left(\frac{d^3y}{dx^3}\right)^2=4AC-5B^2,=3\frac{d^2x}{dy^2}\cdot\frac{d^4x}{dy^4}-5\left(\frac{d^3x}{dy^3}\right)^2=0\ldots.(2).$$

The general equation of a conic, in Cartesian co-ordinates, leads to the pure reciprocant:

$$9\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-45\frac{d^2y}{dx^2}\cdot\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+40\left(\frac{d^3y}{dx^3}\right)^3=A^2D-3ABC+2B^3,$$

$$=9\left(\frac{d^2x}{dy^2}\right)^2\frac{d^5x}{dy^5}-45\frac{d^2x}{dy^2}\cdot\frac{d^3x}{dy^2}\cdot\frac{d^4x}{dy^4}+40\left(\frac{d^3x}{dy^4}\right)=0\ldots.(3),$$ which is appropriately denominated the *Mongian Reciprocant.*

Assume $xy=c$; then $xdy/dx+y=0$, and

$$x=-2\frac{dy}{dx}\bigg/\frac{d^2y}{dx^2}\quad\therefore\ 2\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}-3\left(\frac{d^2y}{dx^2}\right)^2=BT-A^2,=2\frac{dx}{dy}\cdot\frac{d^3x}{dy^3}$$

$$-3\left(\frac{d^2x}{dy^2}\right)^2=0\ldots.(4),$$ which is probably the simplest type of *mixed* reciprocant. From the equation of the hyperbola, $xy+ax+by+c=0$, we can also deduce the *Schwartzian Reciprocant* represented by (4); and after dividing (4) by $(dy/dx)^2$, we have Professor Cayley's *Schwartzian Derivative,*

$$\frac{d^3y/dx^3}{dy/dx}-\frac{3}{2}\left(\frac{d^2y/dx^2}{dy/dx}\right)^2=0,$$ which is directly deducible from $y=(ax+b)/(Ax+B)$, and is of practical use in the solution of differential equations.

From the equation of the circle, $x^2+y^2=r^2$, we have $y(dy/dx)+x=0$;

$$\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}+1=0;\ 3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\bigg/\frac{d^3y}{dx^3}+y=0;\ 3\left(\frac{d^2y}{dx^2}\right)^2\frac{d^3y}{dx^3}$$

$$-3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot\frac{d^4y}{dx^4}+4\left(\frac{d^3y}{dx^3}\right)^2\frac{dy}{dx}=A^2B-2(AC-B^2)T,$$

$$=3\left(\frac{d^2x}{dy^2}\right)^2\frac{d^3x}{dy^3}-3\frac{dx}{dy}\cdot\frac{d^2x}{dy^2}\cdot\frac{d^4x}{dy^4}+4\left(\frac{d^3x}{dy^3}\right)^2\frac{dx}{dy}=0\ldots.(5).$$

The general equation of the circle, after three differentiations, gives

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx}=(1+T^2)B-2A^2T,$$

$$=\left[1+\left(\frac{dx}{dy}\right)^2\right]\frac{d^3x}{dy^3}-3\left(\frac{d^2x}{dy^2}\right)^2\frac{dx}{dy}=0\ldots.(6),$$

The reciprocant represented by (6) may be written

$$1+\left(\frac{dy}{dx}\right)^2-3\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx}\Big/\frac{d^3y}{dx^3}=0\ldots.(\alpha).$$

After differentiating (α), etc., we have the reciprocant represented by (5).

From the equation $x^2+xy+y^2=1$ may be deduced by differentiation, etc.,

$$x,=-\left(\frac{2dy+dx}{dy+2dx}\right)y,=-\left(\frac{2(dy/dx)+1}{dy/dx+2}\right)y\ldots.(a);$$

$$\left(\frac{dy}{dx}+2\right)^2=-\left(2\frac{dy}{dx}+1\right)\left(\frac{dy}{dx}+2\right)\frac{dy}{dx}-3y\frac{d^2y}{dx^2}\ldots.(b);$$

$$3y=-\left[2\left(\frac{dy}{dx}\right)^3+6\left(\frac{dy}{dx}\right)^2+6\left(\frac{dy}{dx}\right)+4\right]\Big/\frac{d^2y}{dx^2}\ldots.(c).$$

$$\therefore 6\left(\frac{dy}{dx}\right)^2\left(\frac{d^2y}{dx^2}\right)^2+15\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx}+6\left(\frac{d^2y}{dx^2}\right)^2-2\left(\frac{dy}{dx}\right)^3\frac{d^3y}{dx^3}-6\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}$$

$$-6\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}-4\frac{d^3y}{dx^3},\ -T^3-\left(\frac{2A^2-3B}{B}\right)T^2-\left(\frac{5A^2-3B}{B}\right)T-2\left(\frac{A^2-B}{B}\right),=0$$

....(7), which is a *cubic* reciprocant. From (6) is deduced $T^2-2(A^2/B)T+1=0$, which is a *quadratic* reciprocant. Transforming (5), we have $T=A^2B/2(AC-B^2);=A^2/B,=5AB/2C,=$ etc., which are *linear* reciprocants. In so far as their number is concerned, the pure reciprocants are like the major planets of the solar system—few; while the mixed reciprocants are like the minor planets of the same system—many.

NOTE.—Since we had the good fortune to be one of Professor Sylvester's students—one of the last and probably the youngest he had—in the Johns Hopkins University, we declare that Dr. Halsted's biographical sketch of that *enthusiastic* mathematical professor and investigator is the acme of appropriateness; and we also declare that all the commendation which that sketch accords to Professor Sylvester, is fully merited by him.

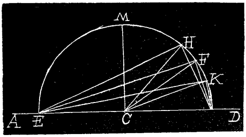
IS THE SUPPLEMENTARY ANGLE FINITE OR NOT?

By Professor JOHN N. LYLE, Ph. D., Westminster College, Fulton, Missouri.

If any individual angle whatever is greater than one right angle and

less than two right angles, is it *finite* and is its supplementary angle, also, *finite* ?

Let EMO be a semicircle and let the lines CH, CF, CK be drawn making the angles ECH, ECF, ECK greater than one right angle and less than two right angles.



Are these angles finite? The answer to this question depends upon the definition of a finite angle.

What, then, is a finite angle? We answer that it is one measured by an arc subtended by a chord that is a *finite* straight line. A *finite* straight line is one that has a beginning and a termination; that is, two ends. Geometrical definitions are *convertible*. Hence, a straight line that has a beginning and termination is *finite*.

Let us examine the angles in the light of these definitions. Each of the chords EH, EF, and EK that subtend the measuring arcs EMH, EMF, and EMK has two ends and by definition is finite. Each of the angles ECH, ECF, and ECK has the distinctive marks of a finite angle and, hence, by definition is *finite*.

Since these angles are less than two right angles and greater than one, the points H, F and K lie on the arc between M and O.

The supplementary angles HCO, FCO, and KCO are measured by the arcs HO, FO and KO whose chords are straight lines having two ends. Hence, by definition both chords and angles are finite. Some say that the angle KCO may become so small that it is *no longer finite* but *an infinitesimal* ! But what is an infinitesimal ? No definition of the term has appeared in any article published thus far in the AMERICAN MATHEMATICAL MONTHLY. Those who persist in using the term should explain with clearness and precision its meaning.

If the supplementary angle KCO is an infinitesimal (whatever that may mean), does the chord of the measuring arc KO retain both of its ends ? If so, is not the chord still finite ? If the chords is finite, is not the measuring arc, also, finite ?

If the measuring arc KO is finite, has not the angle KCO measured by that arc distinctive finite marks? If KCO has distinctive finite marks is it not violative of the logical law of Non-Contradiction to deny this fact by calling the angle *an infinitesimal* ? Furthermore, if the so-called infinitesimal has distinctive finite attributes wherein does it differ from a *finite* quantity?

Bledsoe, in his Philosophy of Mathematics, page 137, quotes Carnot as follows: "Leibnitz who was the first to give rules for the infinitesimal calculus, established it upon the principle that we can take at pleasure, the one for the other, two finite Magnitudes which differ from each other only by a quantity infinitely small.

. . . .

Still it was not free from objections; they reproached Leibnitz (1) with employing the expression infinitely small quantities without having previously defined it; (2) with leaving in doubt, in some sort, whether he regarded his calculus as absolutely rigorous, or as a simple method of approximation."

On page 168 of his Philosophy Bledsoe makes the following quotation from the Marquis de L'Hopital.—"We demand that we can take indifferently the one of two quantities for the other which differ from each other only by a quantity infinitely small: or (what is the same thing) that a quantity which is augmented or diminished by another quantity infinitely small can be considered as remaining the same."

If the difference between the angle sum of Lobatschewsky's triangle and two right angles is an infinitesimal, then according to De L'Hopital's "Demand" Lobatschewsky's angle sum is not less than two right angles but equal thereto.

In Euclid the angle sum of the triangle ECH is two right angles precisely. Hence, the angle sum has no supplementary angle. The assumption that there is a supplementary angle when in truth there is none proves that the words of the hypothesis do not correspond with geometrical fact. Is analytical symbolism properly used when employed to obliterate rather than to represent geometrical data? The hypothesis of *a zero angle* is a relic of logical barbarism whose presence in recent Mathematical literature indicates that the unfittest sometimes survives.

The triangle ECH just constructed is supposed to be in a plane—a surface absolutely without curvature.

If Lobatschewsky's triangle is not in a plane but in a pseudo-spherical surface, it can not be the triangle ECH which is in a plane.

If ECH is a Euclidian triangle, its angle sum is equal to two right angles and it can not at the same time be a Lobatschewsky triangle whose angle sum is less than two right angles.

The assumption that it can is in direct contravention of the logical laws of Non-contradiction and excluded Middle.

Lobatschewsky proved that the sum of the three angles of a rectilineal triangle can not be greater than two right angles. Riemann shows scant respect for Lobatschewsky's demonstration when he maintains that the angle sum of a rectilineal triangle is greater than two right angles.

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

29. Proposed by R. H. YOUNG, West Sunbury, Pennsylvania.

An interest bearing note dated August 1st, 1892, was discounted at 90 days at 8%. The face of the note was \$750, and the proceeds \$759.982. What was the date of discount?

Solution by Professor G. B. M. ZERR, Principal of High School, Staunton, Virgina, and P. S. BERG, Apple Creek, Ohio.

90 days + 3 days grace = 93 days.

Interest on \$1.00 for 93 days at 8% = \$.0206$\frac{2}{3}$.

\$1.00--\$.0206$\frac{2}{3}$=\$.9793$\frac{1}{3}$=proceedr of \$1.00

\$759.982 ÷ .9793$\frac{1}{3}$ = \$776.0197, the amount of the note at the end of the 90 days.

\$776.0197 − \$750. = \$26.0197, interest.

Interest on \$750 for 1 year at 6% =\$45.00.

\$26.0197 ÷ \$45 = 6 months 27 days.

6 months, 28 days — 90 days = 3 months, 28 days.

August 1, 1892 + 3 months 28 days = November 29, 1892.

∴ The note was discounted November 29, 1892.

30. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

While dressing a fifty-cent chicken, a poulterer found a fifty-dollar diamond in the chicken's gizzard. He sold the chicken at a profit of 25cts., changed with good money the conterfeit ten-dollar bill handed him by the unknown purchaser, and realized 50% of the estimated value of the dimond. What per cent. of gain, or loss, did the poulterer make? Suppos the purchaser of the chicken and of the diamond had been one person, what per cent. of gain, or loss, would he have made after selling the diamond for \$25 in good money?

I Solution by the PROPOSER.

The poulterer paid 50 cents for the chicken and gave away \$9.25 in change. His total loss is, therefore, \$9.75. His total gain, the amount realized on the dimond, is \$25.00. Hence the poulterer's net gain is \$15.25, which is 152$\frac{1}{4}$% of the capital invested by him. The unknown purchaser, investing \$25, had aftea his transactions were made, a chicken the market-value of which was 75 cents, and also \$9.25 in good money as change. On his dimond-transaction, he neither gained nor lost. Therefore, the unknown purchaser's total gain is \$10., which is 40% of the capital by him invested during his transactions.

Different interpretations of the conditions of the problem, may lead to different results.

II. **Solution by P. S BERG, Apple Creek, Ohio.**

50 cents, cost of chicken+\$9.25, change given to purchaser of chicken =\$9.75, amount paid by poulterer. 50% of \$50=\$25, amount received by poulterer. \$25—\$9.75=\$15.25 gain. \$15.25÷\$.50×100=3050. Therefore the poulterer gained 3050%.

In the second condition the purchaser makes \$20, and the cost is 75 cents, therefore the gain per cent. is 10÷.75=1333⅓.

An excellent solution with a different result from either of the above was received from *FRANK HORNE*, a former student of Kidder Institute, Columbia, Missouri.

PROBLEMS.

38. **Proposed by J. A. CALDERHEAD, B. Sc., Superintendent of Schools, Limaville, Ohio.**

What must be the thickness of a 36-inch shell, in order that it may weigh 1 ton: supposing a 13-inch shell to weigh 200 pounds, when two inches thick?

39. **Proposed by P. C. CULLEN, Superintendent of Schools, Brady, Nebraska.**

A, *B* and *C* start from same point at same time. *A* north at rate of three miles per hour, *B* east at rate of four miles and *C* west at rate of five miles per hour. *B* at end of two hours starts at such an angle as to intersect *A*. How long after starting must *C* start north-west in order to meet *A* and *B* at common point.?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

27. **Proposed by A. H. BELL, Hillsboro, Illinois. (The problem from H. C. WILKES, Murrayville, West Virginia.)**

An oarsman in rowing a boat down stream 7 miles from *A* to *B* and then back requires 12 minutes longer time, than commencing from *B*, and rowing up and back: the rate of speed for the 1st half of the time is 5 miles per hour, and for the 2nd half of the time is $4\frac{1}{2}$ miles per hour. Required the current.

1. **Solution by H. C. WILKES, Murrayville, West Virginia.**

Let $5+x$=rate down stream; and $5-x$=rate up.

$7+7\left(\frac{5+x}{5-x}\right)=\frac{70}{5-x}$ =length of continuous trip down=in time to trip down and up.

$7+7\left(\frac{4\frac{1}{2}-x}{4\frac{1}{2}+x}\right)=\frac{63}{4\frac{1}{2}+x}$ =length of continuous trip up=in time to trip up and down.

The change of rate in both trips will occur while going up stream and will be in the ratio $5-x:4\frac{1}{2}-x$.

$$\therefore\quad \left(\frac{70}{5-x}\right)\left(\frac{5-x}{9\frac{1}{2}-2x}\right)=\text{distance 1st half of down trip.}$$

$$\left(\frac{63}{4\frac{1}{2}+x}\right)\left(\frac{5-x}{9\frac{1}{2}-2x}\right)=\text{distance 1st half of up trip.}$$

As the difference of time for whole trips is 12 minutes, the difference for $\frac{1}{2}$ time will be 6 minutes, or $\frac{1}{10}$ hour.

Then $\frac{70}{(5+x)(9\frac{1}{2}-2x)}-\frac{63}{(4\frac{1}{2}+x)(9\frac{1}{2}-2x)}=\frac{1}{10}$, weich resolves into the equation $x^3+38x^2+99x-855=0$, whence $x=3$.

II. Solution by A. H. BELL, Hillsboro, Illinois.

Let x=current, rowing down $5+x=a$, $4\frac{1}{2}+x=c$, half time$=\frac{y}{2}=z$.

1st trip rowing up $5-x=b$, $4\frac{1}{2}-x=d$, half time$=\frac{y+\frac{1}{5}}{2}=s$.

Then $\frac{7}{a}$ = time A to B, $s-\frac{7}{a}=\frac{as-7}{a}$ = time left of the 1st half from B to C, and giving the distance $\frac{abs-7b}{a}$; $7-\frac{abs-7b}{a}=\frac{7(a+b)-abs}{a}$ = distance, C to A, and the time $\frac{7(a+b)-abs}{ad}=s=\frac{y+\frac{1}{5}}{2}$, giving $y=\frac{70(a+b)-a(b+d)}{5a(b+d)}$

2nd trip—the 1st half of time from B to D gives the distance bz, distance D to $A=7-bz$, and time $=\frac{7-bz}{d}$, giving the time from A to $B=z-\frac{7-bz}{d}=\frac{z(b+d)-7}{d}$.

The distance equals $\frac{cz(b+d)-7c}{d}=7$ miles, $z=\frac{7(c+d)}{c(b+d)}=\frac{y}{2}$,

$y=\frac{14(c+d)}{c(b+d)}$. Equating the values of y, $c[70(a+b)-a(b+d)]=70a(c+d)$.

Substituting values, $8x^3+38x^2+99x-855=0$, giving $x=3$, with the other values imaginary.

III. Solution by B. F. BURLESON, Oneida Castle, New York, and D. G. DURRANCE, Jr., Camden, New York.

Let x=rate of current; y=distance from B (returning) where the rate of rowing is changed; $2T$=time of rowing from A to B and return; $2t$ = time of rowing from B to A and return; z=dist. from A (going up) where the rate of rowing is changed.

Put $a=7$ miles, $b=5$ miles, and $c=4\frac{1}{2}$ miles. Also put $2d=\frac{1}{5}$ hour, difference in time in making the journeys.

Then, $\dfrac{a}{b+x}+\dfrac{y}{b-x}=T\ldots.(1)$; $\dfrac{a-y}{c-x}=T\ldots.(2)$; $\dfrac{a-z}{b-x}=t\ldots.(3)$;

$\dfrac{z}{c-x}+\dfrac{a}{c+x}=t\ldots.(4)$: and $T=d+t\ldots.(5)$.

From (2), $y=Tx-Tc+a\ldots.(6)$. From (3), $z=tx-tb+a\ldots.(7)$.

Substituteing value of y in (1), $\dfrac{a}{b+x}+\dfrac{Tx-Tc+a}{b-x}=T\ldots.(8)$.

From (5) and (6), $\dfrac{a}{b+x}+\dfrac{dx+tx-cd-ct+a}{b-x}=d+t\ldots.(9)$.

Whence, $t=\dfrac{b^2d-2dx^2-2ab-bdx+bcd+cdx}{2x^2-b^2+bx-bc-cx}$.

The value of z in (4), gives, $\dfrac{tx-tb+a}{c-x}+\dfrac{a}{c+x}=t\ldots.(10)$,

whence $t=\dfrac{2ac}{c^2-2x^2-cx+bx+bc}$. Equating these two values of t, after restoring numerical values, we have $\dfrac{252}{171-8^2+2x}=\dfrac{4x^2+x+1305}{950-10x-40x^2}$, from which $32x^4-326c^2-5301x-\ -16245\ldots.(11)$.

Factoring (11), $(x^2-7\frac{3}{4}x+14\frac{1}{4})(32x^2+248x+1140)=0$.

From first factor, $x=3$ or $4\frac{3}{4}$, and from 2nd $x=-8.25062$ or -7.24938. But of these values only $x=3$ is applieable. By substituting, $2T=5$ hours, $2t=4.8$, $y=3\frac{1}{4}$ miles, and $z=2\frac{1}{5}$ miles.

Also solved by *F. P. MATZ and G. B. M. ZERR.*

28. Proposed by H. W. DRAUGHON, Clinton, Louisiana.

The working capacity of a horse is constant between the ages of a and b years, and decreases at a uniformly accelerated rate from the age of b years to that of c years, becoming 0 at the latter age. If the value of the horse at the age of a years is d, give a formula for finding his value at any subsequent time.

Solution by the PROPOSER.

Let m represent the value of a year's work between the ages of a and b years; then, the value of the total amount of work done between those ages is, $(b-a)\,m\ldots.(1)$.

Let y be the value of the work which the horse could do in one year at the age of $c-1$ years, and let x be any variable time reckoned from the age of c years backward. Then from formulas for uniform acceleration, we have,

$m=(c-b)^2y$. $\therefore$ $y=\dfrac{m}{(c-b)^2}$. The value of the work which could be done at the age of $c-x$ years is, $x^2y=\dfrac{mx^2}{(c-b)^2}$.

$\therefore$ the value of total amount of work done between the ages of b and c years is, $\dfrac{m}{(c-b)^2}\int_0^{c-b} x^2dx=\frac{1}{3}m(c-b)$.

Adding this to value given in (1), we have, $\frac{1}{3}m(c-b)+m(b-a)=d$, since d=value of horse's work from a to c years of age.

$\therefore$ $m=\dfrac{3d}{c-3a+2b}$. For any age, $b-n$, between the ages of a and b years, we have, the value of horse. V=value of work done after that age $=mn+\frac{1}{3}m(c-b)=\dfrac{3d}{c-3a+2b}\,[n+\frac{1}{3}(c-b)]\ldots(A)$.

For any age, $c-n'$, between the ages of b and c years, we have, value of horse,

$$V'=\frac{m}{(c-b)^2}\int_0^{n'}x^2dx=\frac{mn'^3}{3(c-b)^2}=\frac{dn'^3}{(c-b)^2(c-3a+2d)}\ldots(B).$$

Ex. Let us suppose that a horse is worth \$100 at five years of age, and that he begins to weaken at 18 years of age, becoming worthless at 20. Then, $a=5$, $b=18$, $c=20$, and $d=100$. Substituting (A) becomes,

$V=\dfrac{100(3n+2)}{41}$, (A), and (B), becomes, $V'=\dfrac{25n'^3}{41}\ldots(B)$. By substituting for n and n' their successive integral values, we construct the following table:

years	value	years	value	years	value
5	\$100	10	\$63.41	15	\$26.82
6	92.68	11	56.09	16	19.51
7	85.36	12	48.78	17	12.19
8	78.04	13	41.46	18	4.87
9	70.73	14	34.14	19	.60
				20	00

Also solved by *F. P. MATZ, and G. B. M. ZERR.*

29. Sugested by MANSFIELD MERRIMAN, C. E., Ph. D., Professor of Civil Engineering, Lehigh University, South Bethelehem, Pennsylvania.

Solve neatly the equations: $\dfrac{y(1+x^2)}{x(1+y^2)}=a\ldots(1)$, and $\dfrac{y^4(1+x^8)}{x^4(1+y^8)}=b\ldots(2)$.

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Write (1) and (2), respectively,

$$x+\frac{1}{x}=a\left(y+\frac{1}{y}\right)\ldots(3),\text{ and }x^4+\frac{1}{x^4}=b\left(y^4+\frac{1}{y^4}\right)\ldots(4).$$

The second power, and the fourth power, of (3) give respectively, $x^2+\frac{1}{x^2}=a^2\left(y+\frac{1}{y}\right)^2-2\ldots(5)$, and $x^4+\frac{1}{x^4}+4\left(x^2+\frac{1}{x^2}\right)+6$ $=a^4\left(y+\frac{1}{y}\right)^4\ldots(6)$. From (6), by means of (5), we have $x^4+\frac{1}{x^4}+4$ $\left[a^2\left(y+\frac{1}{y}\right)^2-2\right]+6=a^4\left(y+\frac{1}{y}\right)^4\ldots(7)$.

From (4) and (7), $\left(y+\frac{1}{y}\right)^4-\left(\frac{4a^2}{a^4-b}\right)\left(y+\frac{1}{y}\right)^2=-\frac{2}{a^4-b}\ldots(8)$.

$$\therefore\quad y+\frac{1}{y}=\frac{2a^2\pm\sqrt{[2(a^4-b)]}}{a^4-b}=\sqrt{n}\ldots(9).$$

From (9), $y=\frac{1}{2}[\sqrt{n}\pm\sqrt{(n-4)}]\ldots(10)$. From (3), by means of (9), we deduce $x=\frac{1}{2}[a\sqrt{n}\pm\sqrt{(a^2n-4)}]\ldots(11)$.

II. **Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia, and P. S. BERG, Apple Creek, Ohio.**

We may write the expressions as follows:

$\frac{1+x^2}{x}=a\left(\frac{1+y^2}{y}\right)$, and $\frac{1+x^8}{x^4}=b\left(\frac{1+y^8}{y^4}\right)$, or $x+\frac{1}{x}=a\left(y+\frac{1}{y}\right)$, and $x^4+\frac{1}{x^4}$ $=b\left(y^4+\frac{1}{y^4}\right)$. Let $x+\frac{1}{x}=z$, and $y+\frac{1}{y}=v$.

$\therefore\quad z=av\ldots(1)$, $z^4-4z^2+2=b(v^4-4v^2+2)\ldots(2)$.

Substitute (1) in (2), we get, $a^4v^4-4a^2v^2+2=bv^4-4bv^2+2b$.

$\therefore\quad (a^4-b)v^4-4(a^2-b)v^2=2(b-1)$.

$$\therefore\quad v^2=\frac{2(a^2-b)}{a^4-b}\pm\frac{\sqrt{[2(a^2-b)^2+2b(a^4+1)]}}{a^4-b}=n \text{ suppose.}$$

$\therefore\quad v=\pm\sqrt{n}$, $z=\pm a\sqrt{n}$.

$\therefore\ x+\frac{1}{x}=\pm a\sqrt{n}$, or $x^2\pm a\sqrt{n}\,x=-1$, and $y+\frac{1}{y}=\pm\sqrt{n}$, or $y^2\pm\sqrt{n}\,y=-1$

$\therefore\quad x=\pm\frac{1}{2}(a\sqrt{n}+\sqrt{[a^2n-4]})$, $y=\pm\frac{1}{2}(\sqrt{n}+\sqrt{[n-4]})$, which are the correct results, and the same as given by the proposed in a previous number of the MONTHLY.

NOTE—A neat solution to No. 26, Algebra, was received from Jno. B. Fraught, A. B., Instructor in Mathematics, Indiana University, after copy had been sent to the Publishers.

PROBLEMS.

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39. **Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geoditic Survey Office, Washington, D. C.**

Find x, y, z and w from the equations

$$x^4+y^4+z^4+x^2+y^2+z^2=112,$$
$$x^4+z^4+w^4+x^2+z^2+w^2=382,$$
$$x^4+y^4+w^4+x^2+y^2+w^2=294,$$
$$y^4+z^4+w^4+y^2+z^2+w^2=364.$$

40. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find by quadratics all the possible values for x and y in the equations $x^3+y^3=b=35$, ...(1), and $x^2+y^2=a=13$....(2).

41. Proposed by A. H. BELL, Hillsboro, Illinois.

In a right angled triangle there are given, the bisectors of the acute angles. Required the triangle.

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

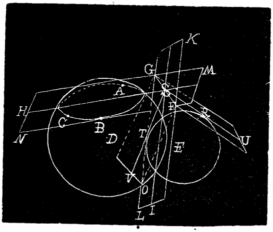
SOLUTIONS TO PROBLEMS.

28. Proposed by Professor HENRY HEATON, M. S., Atlantic, Iowa.

Through three given points to pass two spherical surfaces tangent to a given sphere.

Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let A, B, C be the three given points, and E the given sphere.



Through A, B, C describe any sphere cutting the sphere E. Let D be this sphere. Let the planes through A, B, C and the intersection of E, D intersect in the line FG. Through FG pass the planes FH, GO tangent to E at R, T. Then the spherical surfaces through $ABCR$ and $ABCT$ will be the surfaces required.

For draw the diameter of D perpendicular to the plane ABC, then perpendicular to this diameter and through the centre of the circle made by the plane A, B, C, in the plane A, B, C draw the line SH, also through the centre of the circu-

lar section made by the intersection of D, E and perpendicular to the line joining their centres in the plane KL draw the line ST intersecting the line SH' at S. Draw SR, ST. Then the proof is the same as that for problem 18.

Also solved by Professor H. C. WHITAKER.

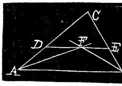
29. Proposed by H. W. HOLYCROSS, Superintendent of Schools, Pottersburg, Ohio.

If the two angles at base of a triangle are bisected; and through the point of meeting of the bisectors a line is drawn parallel to the base, the length of the parallel between the sides is equal to the sum of the segments of the sides between the parallel and the base.

Solution by I. L. BEVERAGE, Monterey, Virginia; I. W. TAYLOR, Berea, Ohio; and the PROPOSER.

Let ABC be any triangle, AF and BF the bisectors of the angles A and B, respectively. Through F draw DE parallel to AB. Then $\angle DFA = \angle FAB$, being alternate interior angles of parallel lines. But $\angle DAF = \angle FAB$, AF being the bisector of $\angle DAB$.

$\therefore \angle DAF = \angle DFA$. $\therefore$ The triangle ADF is isosceles and side AD=side DF. In like manner it may be shown that $FE=EB$.



$\therefore DF+FE$, or $DE, =AD+BE$. Q.E.D.

This problem was solved in slightly different ways by *J. W. Watson, G. B. M Zerr, Cooper D. Schmitt, M. A. Gruber, J. K. Ellwood, R. H. Young, P. S. Berg, John Faught, J. C. Corbin, P. C. Cullen, J. F. W. Scheffer and H. C. Whitaker.*

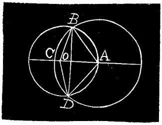
30. Proposed by CHARLES E. MYERS, Canton, Ohio.

A circle containing one acre is cut by another whose center is on the circumference of the given circle, and the area common to both is one-half acre. Find the radius of the cutting circle.

Solution by J. W. WATSON, Middle Creek, Ohio; W. L. HARVEY, Portland, Maine, J. T. FAIRCHILD, Ada, Ohio; P. S. BERG, Apple Creek, Ohio; and JOHN DOLMAN, Jr., Philidelphia, Pennsylvania

Let O be the center of the circle. A the point on the circumference which is the center of the cutting circle, and R its radius.

With A as a center and radius R draw the arc BCD cutting the circumference at B and D. Join AO and produce it to C. Join CB, OB and AB. Put $r=4\sqrt{\left(\frac{10}{\pi}\right)}=7.136$ rods, radius of the first circle. Put $\angle BAO = \angle OBA = \theta$. Then $\angle BOA = \pi - 2\theta$, $\angle BCA = \angle CBA = \frac{1}{2}(\pi-\theta)$.



Sector $BOA = \dfrac{\pi r^2(\pi-2\theta)}{2\pi}$, sector $BAC = \dfrac{\pi R^2\theta}{2\pi}$, $\triangle BAO = \frac{1}{2}Rr\sin\theta$.

Then we have $\dfrac{\pi r^2(\pi-2\theta)}{2\pi} + \dfrac{\pi R^2\theta}{2\pi} - \frac{1}{2}Rr\sin\theta = \dfrac{\pi r^2}{4}$.. But $R=2r\cos\theta$.

Substituting the value of R we have $\pi r^2(\pi-2\theta)+4\pi\theta r^2\cos^2\theta-\pi r^2\sin 2\theta$ $=\frac{1}{2}\pi^2 r^2$ or $2\theta\cos 2\theta-\sin 2\theta=\frac{1}{2}\pi$, from which, $\theta=54°\ 35'\ 39''$.

$\therefore\quad R=2r\cos\theta=8.269265$ rods.

Solutions were also received from *J. F. W. Scheffer, H. C. Whitaker, G. B. M. Zerr, and I. L. Beverage.*

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

22. Proposed by MOSES C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Indiana.

Solve the following Differential Equation:

$$(6x^3+20x^2-2x)\frac{d^2y}{dx^2}-(9x^2+10x+1)\frac{dy}{dx}+(1+9x)y=0.$$

I. Solution by E. B. ESCOTT, Ann Arbor, Michigan.

This can be written $(6x^3+20x^2-2x)\dfrac{d^2y}{dx^2}-(1+9x)(1+x)\dfrac{dy}{dx}+(1+9x)y=0$.

It is evident from inspection that $y=1+x$ is a particular solution.

Assume, therefore, $y=v(1+x)$.

Substituting in the given equation, we have

$$\frac{d^2v}{dx^2}+\left(\frac{2}{1+x}-\frac{9x^2+10x+1}{6x^3+20x^2-2x}\right)\frac{dv}{dx}=0.$$

Let $\dfrac{dv}{dx}=z$, then $\dfrac{d^2v}{dx^2}=\dfrac{dz}{dx}$. $\dfrac{dz}{dx}+\left(\dfrac{2}{1+x}-\dfrac{9x^2+10x+1}{6x^3+20x^2-2x}\right)z=0$.

$$\frac{dz}{z}+\left(\frac{2}{1+x}-\frac{9x^2+10x+1}{6x^3+20x^2-2x}\right)dx=0.$$

Integrating, $z=\dfrac{c(3x^2+10x-1)}{(1+x)^2\sqrt{x}}$ or $dv=\dfrac{c(3x^2+10x-1)}{(1+x)^2\sqrt{x}}dx$.

Integrating, $v=c\sqrt{x}\left(\dfrac{3x-1}{x+1}\right)+c'$. $\therefore\quad y=c(3x-1)\sqrt{x}+c'(1+x)$.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

The equation may be written as follows:

$$\frac{d^2y}{dx^2}-\frac{9x^2+10x+1}{6x^3+20x^2-2x}\frac{dy}{dx}+\frac{9x+1}{6x^3+20x^2-2x}y=0,$$

Let $y=v(x+1)$; then the transformed equation is

$$\frac{d^2v}{dx^2}+\left\{\frac{2}{1+x}-\frac{9x^2+10x+1}{6x^3+20x^2-2x}\right\}\frac{dv}{dx}=0$$

$$\therefore \frac{dv}{dx}(1+x)^2e^{-\int\frac{9x^2+10x+1}{6x^3+20x^2-2x}dx}=A.$$

$$\therefore \frac{dv}{dx}=\frac{A}{(1+x)^2}e^{\int\frac{9x^2+10x+1}{6x^3+20x^2-2x}dx}=\frac{3x^2+10x-1}{x^{\frac{1}{2}}(1+x)^2}A.$$

$$\therefore \quad V=\frac{6x^{\frac{3}{2}}-2x^{\frac{1}{2}}}{1+x}A+B. \qquad \text{But } y=v(x+1), \qquad \therefore y=2x^{\frac{1}{2}}(3x-1)A+(1+x)B,$$

a complete solution, where A and B are arbitrary constants.

Particular solutions are $y=1+x$ and $y=2x^{\frac{1}{2}}(3x-1)$.

III. **Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio, and the PROPOSER.**

Assume $y=A_0x^m\ldots.(1)$; that the given equation becomes

$(6m^2-15m+9)x^{m+1}+(20m^2-30m+1)x^m-(2m^2-m)x^{m-1}=0\ldots.(2)$.

If $m'=m-1\ldots.(3)$, $m'+5=m\ldots.(4)$, $s=1\ldots.(5)$.

Put, then, $y=\Sigma_0^\infty A_rx^{m+r}\ldots.(6)$; and the given equation becomes again

$$\Sigma_0^\infty\{[6(m+r)(m+r-2)-9(m+r)+9]A_rx^{m+r+1}$$
$$+[20(m+r)(m+r-1)-10(m+r)+1]A_rx^{m+r}$$
$$-[2(m+r)(m+r-1)+(m+r)]A_rx^{m+r-1}\}=0\ldots.(7).$$

Putting for r, $r-2$, $r-1$ in the first two terms, (7) becomes

$$\Sigma_0^\infty\{[6(m+r-1)(m+r-3)-9(m+r-2)+9]A_{r-2}$$
$$+[20(m+r-1)(m+r-2)-10(m+r-1)+1]A_{r-1}$$
$$-[2(m+r)(m+r-1)+(m+r)]A_r\}x^{m+r-1}=0\ldots.(8).$$

Making the coefficient of A_r vanish after putting $r=0$, we have $m=0\ldots.(9)$, $m=\frac{1}{2}\ldots.(10)$.

Substituting $m=0$ in the first two terms, and writing $r+2$ for r, $(6r^2-15r+9)A_r+(20r^2+10r-9)A_{r+1}=0\ldots.(11)$. Putting $r=0$, $r=1$ in succession in (11), $A_1=A_0$, $A_2=0$; then one part of the integral is $y'=A_0(1+x)\ldots.(12)$. The like step for $m=\frac{1}{2}$ gives another part of the integral $y''=B_0x^{\frac{1}{2}}(1-3x)\ldots.(13)$.

The whole integral is $y=y'+y''=A_0(1+x)+B_0x^{\frac{1}{2}}(1-3x)\ldots.(14)$.

[This result is also correct since the constant $A'=my-2A$. This has proved a very interesting problem. Prof. Matz sent four different solutions.—ED.]

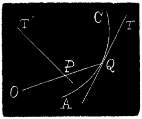
Also solved by *H. C. WHITAKER* and *G. E. WHITE*.

23. **Proposed by W. I. TAYLOR, Baldwin University, Berea, Ohio.**

From a point O situated in the plane of a plane curve, radii vectores are drawn to different points of the curve, and on each one a distance is laid off from O inversely proportional to the length of the radius vector: to determine the tangent at any point of the locus of the points thus obtained. [*Byerly's Diff. Calculus*, p. 177, ex. 1.]

I. Solution by WILLIAM E. BYERLY, Ph. D., Professor of Mathematics in Harvard University, Cambridge, Massachusetts.

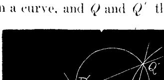
Let AC be an arc of the given curve and let P be the given point of the reciprocal curve at which a tangent is to be drawn.



Draw the radius vector OP and extend it until it meets AC at Q. Draw the tangent QT at Q, and at P draw the line PT' making the angle OPT' equal to the supplement of OQT. Then will PT' be the required tangent. The proof of the correctness of this solution is obvious.

II Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

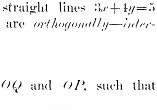
Let P and P' be two contiguous points on a curve, and Q and Q' the *inverse points*; then, since $OP\times OQ=OP'\times OQ'=k^2$, the points P, P', Q' and Q are concyclic; and, since the angles OPT and $OQ'T$ are therefore supplementary, it follows that, in the limit when P' ultimately coincides with P and Q' with Q, the tangents at P and Q make supplementary angles with OPQ.



If the Cartesian Co-ordinates of P be (x,y), and of Q (x',y'); then must $x=k^2x'/(x'^2+y'^2)$, and $y=k^2y'/(x'^2+y'^2)$. Consequently the inverse of the conic $ax^2+2bxy+cy^2=2y$, is the cubic $k^2(ax^2+2bxy+cy^2)=2y(x^2+y^2)$; also, the *inverses* of the lines $x=a$ and $y=b$, are respectively $x^2+y^2=k^2x/a$ and $x^2+y^2=k^2y/b$; and the *inverses* of the straight lines $3x+4y=5$ and $4x-3y=5$, with regard to the co-ordinate origin, are *orthogonally—intersecting* circles.

III. Solution by C. E. WHITE, Trafalgar, Indiana.

If Q' and P' be two points on the vectors OQ and OP, such that $OQ'.OQ=OP'.OP$, it follows that the points Q', Q, P, and P' be on a circle.



Now when P and Q coincide the lines QP and $Q'P'$ become the tangents to the inverse points P and P'. They, also, become the tangents to the circle at these points; hence, the $\triangle SPP'$ is isosceles. Hence whenever the tangent of P is known the tangent at P' may be drawn.

Again if X and Y be the co-ordinates of P' and x and y those of P, then (*Williamson's Calculus*, p. 224) $X=\dfrac{k^2x}{x^2+y^2}$ and $Y=\dfrac{k^2y}{x^2+y^2}$.

Whence $dX=k^2[(y^2-x^2)dx-2xydy]\div(x^2+y^2)^2$ and $dY=k^2[(x^2-y^2)dy-2xydx]\div(x^2+y^2)^2$.

$$\text{Hence } \frac{dX}{dY}=\frac{(x^2-y^2)dy-2xydx}{(y^2-x^2)dx-2xydy}.$$

Hence, if x' and y' be the co-ordinates of a point on the curve, the tangent at its inverse point may be

$$\text{written } \left(y-\frac{k^2y'}{x'^2+y'^2}\right)=\frac{(x'^2-y'^2)dy'-2x'y'dx'}{(y'^2-x'^2)dx'-2x'y'dy'}\left(x-\frac{k^2x'}{x'^2+y'^2}\right).$$

Also solved by *G. B. M. ZERR*.

PROBLEMS.

30. **Proposed by E. W. NICHOLS, Professor of Mathematics in the Virginia Military Institute, Lexington, Virginia.**

Given the cardioid $r=a(1-\cos\theta)$; find the area of its circumscribing square formed by tangents making angles of 45° with its axis.

31. **Proposed by E. B. ESCOTT, Ann Arbor, Michigan.**

Through a point O on the produced diameter AB of a simicircle draw a secant ORR', so that the quadrilateral $ABRR'$ inscribed in the semicircle shall be a maximum. Prove that in this case, the projection of RR' on AB is equal in length to the radius of the circle. [*Williamson's Diff. Calculus. 7th edition, p. 189, Ex. 25.*]

MECHANICS.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

12. **Proposed by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.**

A horizontal table without weight is supported on three points A, B, and C. A weight W is laid upon the table, at a point G. If $AG=a$, $BG=b$, $CG=c$, $\angle AGB=\beta$, and $\angle AGC=\nu$, find the pressures upon A, B, and C.

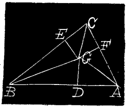
I. **Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and L. B. FRAKER, Weston, Ohio.**

From the problem, we have $AB=\sqrt{(a^2+b^2-2ab\cos\beta)},=m$. Simi-

larly $AC=n$, and $CB=p$. Put $\angle AGD=\phi$, $\angle CGF=\psi$, and $\angle BGE=\omega$; then, obviously, $a\sin\phi+b\sin(\beta-\phi)=m$....(1),

$c\sin\psi+a\sin(\nu-\psi)=n$....(2),

and $b\sin\omega+c\sin[2\pi-(\beta+\nu-\omega)]=p$....(3).



Knowing the values of ϕ, ψ, and ω, from (1), (2), and (3), respectively, we obtain for the presures upon A, B, and C.

$\mathbf{P}_A=a[\sin\phi+\sin(\nu-\psi)]W$....(4),

$\mathbf{P}_B=b[\sin\omega+\sin(\beta-\phi)]W$....(5),

and $\mathbf{P}_C=c[\sin\psi-\sin(\beta+\nu-\omega)]W$....(6).

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O, Mississippi, A. H. BELL, Hillsboro, Illinois; and the PROPOSER.

Let the pressures at A, B, and C be represented by these letters.

Taking moments about AG, $b\sin\beta.B=c\sin\gamma.C$.

Taking moments about CG, $a\sin\gamma.A=-b\sin(\beta+\gamma)B$.

Also, $A+B+C=W$. From these,

$$B=\frac{\sin\gamma}{b}W\div\left(\frac{-\sin(\beta+\gamma)}{a}+\frac{\sin\gamma}{b}+\frac{\sin\beta}{c}\right).$$

Denoting the quantity in the parenthesis by K, we have, from considerations of symmetry, $A=\dfrac{-\sin(\beta+\gamma)}{aK}W$, $C=\dfrac{\sin\beta}{cK}W$.

Excellent solutions to this problem were received from *G. B. M. ZERR, and P. H. PHILBRICK*

13. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

A man, horse and buggy are going around a circular race course at a 2:40 gait. If the whole outfit weighs 1500 lbs, radius of course is 500 feet and track is inclined so that pressure is equal upon the wheels, find the pressure on the ground due to whole weight.

Solution by the PROPOSER.

In the figure, let C, the bob of the conical pendulum, represent the whole weight of man, horse and buggy$=M$. Let r=radius of curve, V=velocity per second, f=centrifugal force. Then $CG=m$, $CE=f$. Resolve CG into its components CH perpendicular to, and CL parallel to AC; also, CE into its components $CD=CH$ perpendicular to, and CK parallel to AC.

Then pressure$=P=m\cos\theta+f\sin\theta$, where $\theta=OAC$.

But $f=\dfrac{mv^2}{gr}$, and $v^2=\dfrac{gr^2}{h}$, where $h=AO$, $r=OC$.

$$\sin\theta=\frac{r}{\sqrt{(r^2+h^2)}},\ \cos\theta=\frac{h}{\sqrt{(r^2+h^2)}}.\quad\therefore\quad P=\frac{m\sqrt{(r^2+h^2)}}{h}.$$

Now let abc be a section of the road bed; then abc is similar to OAC.

$ab=a$, $bc=e$, $ac=w$. Then $\dfrac{\sqrt{(r^2+h^2)}}{h}=\dfrac{a}{w}=\dfrac{a}{a\cos\theta}$. From the triangle

CEG we get $\tan\theta=\frac{v^2}{gr}$, $\cos\theta=\frac{gr}{\sqrt{(g^2r^2+v^4)}}$. $\therefore$ $\sqrt{\frac{(r^2+h^2)}{h^2}}=\sec\theta$

$=\frac{\sqrt{(g^2r^2+v^4)}}{gr}$. $\therefore$ $P=\frac{m\sqrt{(g^2r^2+v^4)}}{gr}$, but $m=1500$ pounds, $r=500$ feet, $v=33$ feet per second, $g=32$ feet. Substituting we get, $P=1503.465$ pounds.

NOTE.—Professor Zerr constructed a diagram to accompany his solution but as the solution is sufficently lucid without, we did not have a diagram made.

II. Solution by WILLIAM HOOVER, A. M., Ph. D., Athens, Ohio.

Let 500 feet $=r=$ the radius of the curve described by the center of gravity of the carriage considered horizontal, $a=$ the vertical distance of the center of gravity from the road, $2b=$ the width of the road, $v=$ the velocity, and $mg=$ the weight of the "outfit."

The resultant of the horizontal centrifugal force and the weight of the outfit must be perpendicular to the road, and if $\theta=$ the inclination of the road, we have, taking moments about a point in the outer circumference of the road,

$(a\cos\theta-b\sin\theta)\frac{mv^2}{r}=mg(a\sin\theta+b\cos\theta)\ldots.(1)$,

whence $\tan\theta=\frac{av^2-bgr}{bv^2+agr}\ldots.(2)$.

No value for b seems to be contemplated in the statement of the problem. Putting $b=0$ in (2), $\tan\theta=\frac{v^2}{gr}\ldots.(3)$.

The whole pressure $=\frac{mv^2}{r}\sin\theta+mg\cos\theta\ldots.(4)$. The rest is only the numerical application of (4).

This problem was also solved by *Professors MATZ, PHILBRICK and HUME.*

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

13. **Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.**

It is required to find four numbers the sum of whose fourth powers is a square number.

Solution by the PROPOSER.

Let x, $x-ay$, $x-by$, $x-cy$ denote the four numbers required; then must $x^4+(x-ay)^4+(x-by)^4+(x-cy)^4=\square$ ....(1).

Expanding and arranging the terms according to the terms of x, we have $4x^4-4(a+b+c)x^3y+6(a^2+b^2+c^2)x^2y^2-4(a^3+b^3+c^3)xy^3+(a^4+b^4+c^4)y^4=\square$ ....(2); or, putting $a+b+c=m$, $a^2+b^2+c^2=n$, $a^3+b^3+c^3=p$,

$a^4+b^4+c^4=q$, $4x^4-4mx^3y+6nx^2y^2-4pxy^3+qy^4=\square$ ....(4).

Putting $(4)=[2x^2-mxy+\frac{1}{4}(6n-m^2)y^2]^2$, expanding and reducing, we find

$$\frac{x}{y}=\frac{(6n-m^2)^2-16q}{8m(6n-m^2)-64p},$$

$$=\frac{[6(a^2+b^2+c^2)-(a+b+c)^2]^2-16(a^4+b^4+c^4)}{8(a+b+c)[6(a^2+b^2+c^2)-(a+b+c)^2]-64(a^3+b^3+c^3)}.$$

If $a=5$, $b=2$, $c=1$, then $\frac{x}{y}=\frac{199}{-72}$, and we may take $x=199, y=-72$.

These values give the numbers 199, 271, 343, and 559.

$\therefore\quad 199^4+271^4+343^4+559^4=344162^2$.

14. Proposed by SYLVESTER ROBINS, North Branch Depot, New Jersey.

Find initial terms in each of three infinite series of prime, integral, rational, scalene triangles, where 9 shall be the base, and the other two sides of every term shall have a constant difference.

Solution by ARTEMAS MARTIN, LL D, U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $x+\frac{1}{2}d$ and $x-\frac{1}{2}d$ be "the other two sides;" then the area of the triangle is $\sqrt{[(x+4\frac{1}{2})(4\frac{1}{2}-\frac{1}{2}d)(4\frac{1}{2}+\frac{1}{2}d)(x-4\frac{1}{2})]}$, which must be rational, or $(20\frac{1}{4}-\frac{1}{4}d^2)(x^2-20\frac{1}{4})=\square$ ....(1).

Put $20\frac{1}{4}-\frac{1}{4}d^2=c^2e$, then (1) becomes $e(x^2-20\frac{1}{4})=\square$ ....(2).

Put $e(x^2-20\frac{1}{4})=\frac{p^2}{q^2}(x+4\frac{1}{2})^2$, then we get $x=\frac{4\frac{1}{2}(eq^2+p^2)}{eq^2-p^2}$;

$$\therefore\quad x+\tfrac{1}{2}d=\frac{(4\frac{1}{2}+\frac{1}{2}d)eq^2+(4\frac{1}{2}-\frac{1}{2}d)p^2}{eq^2-p^2},\quad x-\tfrac{1}{2}d=\frac{(4\frac{1}{2}-\frac{1}{2}d)eq^2+(4\frac{1}{2}+\frac{1}{2}d)p^2}{eq^2-p^2}.$$

Now eq^2-p^2 may have any value, positive or negative, that will exactly divide the numerators of the expressions for $x+\frac{1}{2}d$ and $x-\frac{1}{2}d$.

It is plain that d must be an odd number, and it must be less than 9, and therefore can not be greater than 7.

I.—Take $d=1$, then $e=5$, and $5q^2\backsim p=1$ will satisfy this case. The least values of p and q are $p=2$, $q=1$, which give 40 and 41 for "the other two sides." Therefore the sides of the first triangle of the series having 9 for bases and a constant difference of unity between the other two sides are 9, 40, 41.

II.—When $d=3$, we have the numerator of $x+\frac{1}{2}d=6eq^2+3p^2=3(2eq^2+p^2)$, and the numerator of $x-\frac{1}{2}d=3eq^2+6p^2=3(eq^2+2p^2)$, both of which are always divisible by 3 whatever the valaes of p and q; hence there can not

be "prime, integral, rational, triangles" having 9 for base and 3 for difference of the other two sides.

III.—When $d=5$, then $e=14$. In this case take $14q^2 \backsim p^2=2$; the least values of p and q are $p=4$, $q=1$, and we find the sides of the first triangle in this case to be 9, 60, 65.

IV.—When $d=7$, then $e=2$, and $2q^2 \backsim p^2=1$ will give the required results. Taking $p=1$, $q=1$, we find the sides of the first triangle to be 9, 10, 17.

II. Solution by the PROPOSER.

I. Let 9, $x-\frac{1}{2}$, $x+\frac{1}{2}$ be sides of a rational $\triangle$.

Then $(x+4\frac{1}{2})(x-4\frac{1}{2})5\times4=\square$ of area. $5(x^2-20\frac{1}{4})=\square$.

Key$=\sqrt{5}=\frac{2}{1}, \frac{9}{4}, \frac{38}{17}, \frac{161}{72}, \frac{682}{305}, \frac{2889}{1292}$, &c.

$5\times4^2+1^2=81$,	I. $\triangle$	9	40	41,
$5\times17^2+2^2=1449$,	II.	9	724	725,
$5\times72^2+9^2=26001$,	III.	9	13000	13001,
$5\times305^2+38^2=466569$,	IV.	9	233284	233285,
$5\times1292^2+161^2=8372241$,	V.	9	4186120	4186121, &c.

II. Let 9, $x-2\frac{1}{2}$, $x+2\frac{1}{2}$=sides of $\triangle$. $(x+4\frac{1}{2})(x-4\frac{1}{2})7\times2=\square$.

$14(x^2-20\frac{1}{4})=\square$. Key$=\sqrt{14}=\frac{3}{1}, \frac{4}{1}, \frac{11}{3}, \frac{15}{4} \cdot\cdot \frac{449}{120} \cdot\cdot \frac{13455}{3596} \cdot\cdot \frac{403201}{107760}$, &c.

$9\times15=135$,	I. $\triangle$	9	65	70,
$9\times449=4041$,	II.	9	2018	2023,
$9\times13455=121095$,	III.	9	60545	60550,
$9\times403201=3628809$,	IV.	9	1814402	1814407,
$9\times12082575=108743175$,	V.	9	54371585	54371590, &c.

III. Let 9, $x-3\frac{1}{2}$ and $x+3\frac{1}{2}$=sides of $\triangle$.

$(x+4\frac{1}{2})(x-4\frac{1}{2})\ 8\times1=\square$. $2(x^2-20\frac{1}{4})=\square$.

Key$=\sqrt{2}=\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}$, &c.

$5^2+2=27$,	I. $\triangle$	9	10	17,
$12^2+3^2+0=153$,	II.	9	73	80,
$29^2+7^2+1^2=891$,	III.	9	442	449,
$70^2+17^2+2^2=5193$,	IV.	9	2593	2600,
$169^2+41^2+5^2=30267$,	V.	9	15130	15137,
$408^2+99^2+12^2=176409$,	VI.	9	88201	88208, &c.

P. S. BERG refers to the proposer's article on pp. 262-3 of August MONTHLY.

PROBLEMS.

20. Proposed by G. B. M. ZERR, Principal of High School, Staunton, Virginia.

Find two integral numbers, whose sum, difference, and difference of their squares shall each be a square, cube and fourth power.

21. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland,

Find (1) nine positive *integral numbers* in arithmetical progression the sum of whose squares is a *square number*: and (2) find nine integral *square numbers* whose sum is a *square number*.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

10. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Upon a surface one foot square, a coin one inch in diameter is thrown at random: what is the chance the coin *touches* or *intersects* both diagonals?

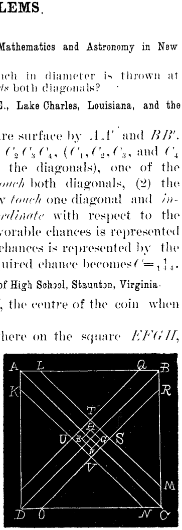
I. Solution by Professor P. H. PHILBRICK, M. Sc., C. E., Lake Charles, Louisiana, and the PROPOSER.

Represent the diagonals of the given square surface by AA' and BB'. If the center of the coin fall within the square $C_1C_2C_3C_4$, (C_1, C_2, C_3, and C_4 being the center of the coin when it is tangent to the diagonals), one of the following conditions is fulfiled: (1) the coin may *touch* both diagonals, (2) the coin may *intersect* both diagonals, (3) the coin may *touch* one diagonal and *intersect* the other. Since these conditions are *co-ordinate* with respect to the conditions of the problem, the total number of favorable chances is represented by the square $C_1C_2C_3C_4$. The total number of chances is represented by the area of the given square $ABA'B'$. Hence the required chance becomes $C=\frac{1}{144}$.

II. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $ABCD$ be the square. E, F, G, H, the centre of the coin when it is tangent to both diagonals.

Then if the centre of the coin falls any where on the square $EFGH$, it will either touch or fall upon both diagonals. If the center falls any where on the area $AKUPDOVNCMSRBQTLA$, except when upon the square $EFGH$, it will touch or fall upon one diagonal. If the centre falls any where upon the areas KUP, OVN, MSR, QTL it will neither touch or fall upon a diagonal. Let A=area of $ABCD$=144 sq. inches.



a=area of $EFGH$=1 sq. inch.

b=area of $AKUPDOVNCMSRBQTLA$ $-$area $EFGH$. $\therefore$ $b=(48\sqrt{2}-9)$ sq. inches.

c=area $(KUP+OVN\times MSR+QTL)=(152-48\sqrt{2})$ sq. inches.

$\therefore\ \ p$=chance that it touches or falls on both diagonals$=\frac{a}{A}=\frac{1}{144}$.

$\therefore\ \ p_1$=chance that it touches or falls on one diagonal$=\frac{b}{A}=\frac{48\sqrt{2}-9}{144}$.

$\therefore\ \ p_2$=chance that it does not touch or fall on a diagonal$=\frac{c}{A}=\frac{152-48\sqrt{2}}{144}$.

$\therefore\ \ p+p_1+p_2=1$.

NOTE.—This problem was solved with different results by H. W. Draughon, Hon. Sosiah Drummond and ———. Professor Draughon and ———'s result is $\frac{1}{121}$. They suppose that the surface of the coin must be entirly on the given square, thus reducing the area of the surface upon which the centre of the coin may fall by a half-inch strip on each side.

Mr. D. consider the surface as though it were the bottom of a box. In this case, the area on which the coin could fall is $\left[144-\left(1-\frac{\pi}{4}\right)\right]$ square inches. Then the probability required is $\frac{4}{572-\pi}$. Each of these three results is right when viewed from the stand-point of its author, but we are doubtful whether the result $\frac{1}{121}$ can be harmonized with the theory of probability.

11. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Find the average area of a triangle formed by joining a corner of a cube with any two points within the cube.

Solution by F. P. MATZ, M. Sc., Ph. D, Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland. and Professor G B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Take a lower corner O of the cube as the origin of co-ordinates, and let P_1 and P_2 be any two points taken at random within the cube. Make (x, y, z) and (u, v, w) the Cartesian triple co-ordinates of P_1 and P_2 respectively; then will $OP_1=\sqrt{(x^2+y^2+z^2)}$, and $OP_2=\sqrt{(u^2+v^2+w^2)}$.

Considering P_1 the *remoter* point with respect to the origin of co-ordinates, we have $P_1P_2=\sqrt{[(x-u)^2+(y-v)^2+(z-w)^2]}$; and consequently,

$$\cos\angle P_1OP_2=\cos\phi=\frac{(OP_1)^2-(OP_2)^2-(P_1P_2)^2}{2(OP_1)(OP_2)},=\frac{ux+vy+wz}{2(OP_1)(OP_2)}.$$

$$\therefore\ \ \sin\phi=\sqrt{(1-\cos^2\phi)}=\frac{\sqrt{[(vx-uy)^2+(wx-uz)^2+(wy-vz)^2]}}{2(OP_1)(OP_2)},\text{ and}$$

$$\triangle P_1OP_2=A=\tfrac{1}{2}(OP_1)(OP_2)\sin\phi=\tfrac{1}{2}\sqrt{[(vx-uy)^2+(wx-uz)^2+(wy-vz)^2]}.$$

Hence the required average area becomes

$$\mathbf{A}=\frac{\int_0^s\int_0^s\int_0^s\int_0^x\int_0^y\int_0^z A\,dxdydzdudvdw}{\int_0^s\int_0^s\int_0^s\int_0^x\int_0^y\int_0^z dxdydzdudvdw},$$

in which s represents a side of the given cube. The labor required to perform the indicated integrations is *enormous*—enough to discourage the most enthusiastic mathematical genius.

NOTE—Since the parenthetical expressions in $\triangle P_1P_2O=\frac{1}{2}\surd[(vx-uy)^2+(wx-uz)^2+(wy-vz)^2]$ represent respectively 2(Area of the projections of of $\triangle OP_2P_1$) on the co ordinate planes XY, ZX, the result of problem 2, Average and Probability in April No. of the MONTHLY, substituted in these expressions gives $\triangle P_1P_2O=\frac{13}{108}a^2\surd3$ as the required average area.—MATZ.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

11. Proposed by CHARLES E. MYERS, Canton Ohio

"Assuming the earth's orbit to be a circle, if a comet move in a parabola around the sun and in the plane of the earth's orbit, show that the comet cannot remain within the earth's orbit longer than 78 days."

I. Solution by WILLIAM HOOVER, A. M Ph. D., Professor of Mathematics and Astronomy in the Ohio University, Athens, Ohio

Let $4a$=the latus rectum of the comet's orbit, ρ=its distance at any time from the sun, p=the perpendicular from the sun upon a tangent to the comet's path, F=the attractive force of the sun, h=the double area described by ρ in a unit of time, and θ the angular co-ordinate corresponding to ρ. Then $F=\dfrac{h^2}{p^3}\dfrac{dp}{d\rho}$ ....(1), and from the parabola $\dfrac{2}{p^3}\dfrac{dp}{d\rho}=\dfrac{1}{a\rho^2}$; $\therefore$ $F=\dfrac{h^2}{2a\rho^2}$ ....(2).

Let $F=\phi$, when $\rho=1$; then $h=\surd(2a\phi)$ ....(3), and $F=\dfrac{\phi}{\rho^2}$....(4).

If r=the radius of the earth's orbit, and v=the velocity of the earth, $v^2=rF=\dfrac{\phi}{r}$, or $v=\sqrt{\dfrac{\phi}{r}}$. Then $\theta r\div v=\dfrac{\theta(r)^{\frac{3}{2}}}{\surd\phi}$=the time required for the earth to describe the arc subtending the angle θ at the sun....(e).

For the comet, $dt=\dfrac{\rho^2d\theta}{h}$ ....(5), and from the parabola,

$\rho=\dfrac{2a}{1+\cos\theta}=\dfrac{a}{\cos^2\frac{1}{2}\theta}$....(6). This gives $\rho^2d\theta=\dfrac{a^2d\theta}{\cos^4\frac{1}{2}\theta}$, and then (5) gives

$$t=\frac{2a^2}{h}\int\frac{d\theta}{\cos^4\frac{1}{2}\theta}=\frac{2a^2}{\surd2a\phi}\left\{\tan\tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta\right\}....(7).$$

The circle $\rho=r$ intersects (6) in a point given by $\cos\frac{1}{2}\theta=\sqrt{\frac{a}{r}}$ or

$\tan\frac{1}{2}\theta=\sqrt{\frac{r-a}{a}}$; we then have $2t=2a^{\frac{3}{2}}\sqrt{\frac{2}{\phi}}\left\{\sqrt{\frac{r-a}{a}}+\frac{1}{3}\sqrt{\frac{(r-a)^3}{a^3}}\right\}$

$=\frac{2}{3}\sqrt{\frac{2}{\phi}}(r+2a)\sqrt{(r-a)}\ldots.(8)$. This must be a maximum.

Equating $\frac{dt}{dr}$ to zero and solving for r, we find $r=2a$.

$\therefore\quad \theta=\frac{\pi}{2}$, and (e) becomes $\frac{2\pi a}{1}\sqrt{\frac{2a}{\phi}}$ and (8), $\frac{4}{3}a\sqrt{\frac{2a}{\phi}}$. These give for the greatest part of the earth's year during which a parabolic comet can remain in the earth's orbit is $\frac{2a}{3\pi}$, or about 78 days.

II. Solution by G. B. M ZERR, A. M. Principal of High School, Staunton, Virgina.

Let $r=a$ be the equation to the earth's orbit.

$r=\frac{2d}{1+\cos\theta}$ be the equation to the comet's orbit.

Also $u=\frac{1}{r}$, for the circle $\frac{du}{d\theta}=\frac{d^2u}{d\theta^2}=0$, for the parabola $\frac{du}{d\theta}$

$=-\frac{\sin\theta}{2d}$, $\frac{d^2u}{d\theta^2}=-\frac{\cos\theta}{2d}$, but P=force of attraction$=h^2u^2\left\{\frac{d^2u}{d\theta^2}+u\right\}$.

$\therefore\quad P=\frac{h^2u^2}{a}=\frac{h^2}{a^3}$ for circle. $P=h^2u^2\left\{\frac{1+\cos\theta}{2d}-\frac{\cos\theta}{2d}\right\}=\frac{h^2}{2dr^2}$

for the parabola at the point of intersection of the two curves $r=a$ and the values of P for each are equal.

$\therefore\quad \frac{h^3}{a^2}=\frac{h^2}{2dr^2}=\frac{h^2}{2da^2}$, $\therefore\quad a=2d$. Also at the intersection

$r=\frac{2d}{1+\cos\theta}=a$, but $a=2d$. $\therefore\quad \cos\theta=0$, and $\theta=\frac{\pi}{2}$.

The tirms of describing any given angle is obtained from the formula $r^2\frac{d\theta}{dt}=h$. $\therefore\ dt=\frac{r^2}{h}d\theta$, we found θ above to be $\frac{\pi}{2}$ measured from the vertex of the parabola. Hence the time the comet is within the earth's orbit is

$t=\frac{2a^2}{h}\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1+\cos\theta)^2}=\frac{4a^2}{3h}=\frac{4}{3}\sqrt{\frac{a^3}{\mu}}$ where μ=absolute force. The periodic time for the earth is $t=4\int_0^{\frac{\pi}{2}}\frac{r^2}{h}d\theta=\frac{4a^2}{h}\int_0^{\frac{\pi}{2}}d\theta=\frac{2\pi a^2}{h}=2\pi\sqrt{\frac{a^3}{\mu}}=1$ year.

Let x=time the comet is within the earth's orbit;

then $2\pi\sqrt{\dfrac{a^3}{\mu}} : \dfrac{4}{3}\sqrt{\dfrac{a^3}{\mu}} = 1$ year : x. $\therefore\ x=\left(\dfrac{2}{3\pi}\right)^{th}$ part of a year

$= \dfrac{2}{3\pi}\times 365\frac{1}{4}$ days $=77.208+$ days.

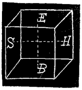
PROBLEMS.

20. **Proposed by SAMUEL HART WRIGHT, M. D., M. A., Ph. D., Penn Yan, Yates County, New York.**

When does The Dog-Star and the Sun rise together in Latitude 42° 30′ North=λ? Given the R. A. of Sirius=6 h. 40 m. 30 sec. and its Dec.= 16° 33′ 56″ S.

21. **Proposed by J. K. ELLWOOD, A. M , Principal of Colfax School, Pittsburg, Pennsylvania.**

Solve (if possible) the following: A cube whose edge is a feet revolves on both axes—EB and SH—at the same number of revolutions per minute.



What is the volume of the figure generated,(a) when the center of the cube remains in one place, (b) when the center of the cube moves b feet in a straight line in a minute?

QUERIES AND INFORMATION

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

Notes on Counsellor Dolman's Remarks In April Number.

By Professor John N. Lyle.

[Received May, 1894.]

Says Counsellor Dolman—"According to Lobatschewsky the angle-sum of a rectilineal triangle decreases as the area of the triangle increases, but is always less than two right angles."

What is a *rectilineal* triangle? The answer is, one whose sides are straight lines. "A triangle can be formed of three straight lines joining any three points." As three points determine the position of a plane, the surface of a rectilineal triangle is a plane.

The triangle whose angle-sum is assumed to be less than two right angles is according to Lobatschewsky *rectilineal*. But Counsellor Dolman tells us that "His (Lobatschewsky's) straight line is not, (it is true), really straight." How can a triangle be called rectilineal whose sides are avowedly not straight? There is palpable contradiction here. If so, what becomes of the boasted consistency of Lobatschewsky's geometry?

Once introduce germinal error and the disease spreads.

The denial of the truth of Euclid's axiom 12 as a geometrical proposition has according to Counsellor Dolman's own showing led to wide spread demoralization in geometry. As a result of that denial we find that the angle-sum of a triangle is changed, that planes are warped, that straight lines are not "really straight", and that even space itself is transmitted into something said to be pseudo-spherical, different from the space which we know and in which all dwell whether they be common sense, Euclidian mathematicians or hyper-space, anti-Euclidian geniuses.

Counsellor Dolman thinks that I misapprehend the meaning of Lobatschewsky. Is he quite sure that Lobatschewsky would have admitted that "his geometry does not apply to the plane, nor to space as we know it"? Counsellor Dolman applied the adjectives "finite" and "infinite" to straight lines but does not tell us what he means by them. He objects to the definition that a finite straight line is one that has two ends. Why does he object? Has he some "infinite straight lines, each of which has two ends, that he wishes to exhibit to the court?

The Counsellor complains that I confound "infinite" and "boundless". Will he kindly explain the difference between an "infinite" and a "boundless" straight line? Riemann makes the distinction between the "infinite" and "boundless" but it is unsatisfactory. I understand that Riemann accepts the Leibnitzian hierarcy of infinite quantities, some of which are an infinite number of times larger than others. Is that Counsellor Dolman's view also? Camplaint was made against Leibnitz and his school that they would not define what they meant by infinitely great and infinitely small quantities. Can we not always find *finite* quantity less than assigned *finite* quantity? Is there not a deal of juggling with the phrases "infiitely small" and "less than any assignable quantity"?

If as the Counsellor informs us "Lobatschewsky's straight line is not really straight, but is the shortest distance between two points, and lying wholly in the given space," it would seem that not only is Euclid's postulate 1 eliminated from Lobatschewsky's space but also his straight line itself. The shortest distance between two points in Lobatschewsky's space, we are told, is not really a straight line. Can a really straight line be drawn in Lobatschewsky's space? If not, what becomes of postulate 1 of Euclid's elements and of Lobatschewsky's *rectilineal* triangle?

I would like to see a Lobatschewsky triangle before I die but if it is impossible to materialize the nondescript in "space as we know it" I suppose that I will have to forego the pleasure.

I will close by thanking Counsellor Dolman for his criticisms. I am quite as fond of having my errors pointed out as the non-Euclidians are of having theirs exposed. My honest opinion, however, is that the Counsellor defended more errors than he corrected.

A Reply to Mr. Draughon. By H. C. Whitaker.

[Received in September, 1894.]

In reply to Mr. Draughon with reference to the equation $\sqrt{(x+4)} - \sqrt{(x-4)} = 4$, I cannot agree with him that the minus sign before the radical $\sqrt{(x-4)}$ must necessarily be one of operation and that the direction of $\sqrt{(x-4)}$ is still ambiguous. I regard the sign as a combination of the two. To illustrate. In the early portions of Algebra, it is usual to give problems such as $9+(-7)=?$ $9-(+7)=?$ $9-(-7)=?$; later in the subject, in enunciating exercises, these operations are supposed to be performed, and the single sign is deemed sufficient; thus, instead of saying $9+(-7)$ or $9-(+7)$, simply $9-7$ is used and I think no ambiguity arises from its use.

Now if Mr. Draughon means to imply that mathematical convention has not established the usage that in a polynomial a single sign before a radical of the second degree is not sufficient to indicate a particular root, I could refer him I presume to a hundred and probably a thousand references in the works of our best mathematicians where exactly that assumption is made. This idea is fundamental in the problem of rationalizing the denominator of a surd fraction, and it is the basis of the discussion of symmetry in plotting equations. Let me quote a few words from Olney's University Algebra, page 131, where the discussion of the solution of $x^2+px=q$ is taken up, the value of x having been found to be $-\frac{p}{2} \pm\sqrt{\frac{p^2}{4}+q}$: "When p and q are both positive, $-\frac{p}{2}+\sqrt{\frac{p^2}{4}+q}$ is positive and $-\frac{p}{2}-\sqrt{\frac{p^2}{4}+q}$ is negative. When p is negative and q positive, if we take the plus sign of the radical, x is positive, but if we take the minus sign, x is negative" and so on. I believe that Mr Draughon himself would calculate $3+\sqrt{2}$ to be $4.4142+$ and would be surprised if he were told the value was 1.5858 and would also be surprised if he were told that the expression as above written has no meaning, and that in order to make it intelligible, it must be written either $3+ +\sqrt{2}$ or $3- -\sqrt{2}$. For myself, I see no advantage in using the two signs, and hope that, if Mr. Draughon advocates their use, the mathematical world will not adopt his suggestion. No discourtesy to Mr D. intended.

But even adopting this suggestion, we are as badly off as ever in answering L. B's question, if he amends it to read $+\sqrt{(x+4)}+-\sqrt{(x-4)}=4$, where the sign before the radicals now clearly indicatee direction.

The total dissimilarity between this surd equation and the equation $x^2+2x=3$, which Mr. Draughon compares it to, can be seen by changing the signs of the various terms of the quadratic; in every case (3 more besides the given equation) new roots will be produced, the equations obtained being wholly dissimilar to the original equation; but no amount of changes of sign of the terms of $\sqrt{(x+a)}-\sqrt{(x-a)}-a=0$ can produce an equation which has more than one quarter of a chance of having one root, the product of the given equation and the equations obtained by changing signs giving only an equation of the first degree, while the product of $x^2+2x-3=0$, $x^2+2x+3=0$, $x^2-2x-3=0$ and $x^2-2x+3=0$, give an equation of the eighth degree.

It is undoubtedly true that by a hocus-pocus Mr. Draughon seems to make $+\sqrt{(x-4)}=-1$, but that was by doing exactly what I said was a wrong operation. Take his equation $x^2=3-2x$; square it; among the roots of this last equation will be found $x=1\pm\sqrt{-2}$, which will of course prove if we assume that x^2 must equal $3-2x$; as a matter of fact however for this value of x x^2 must $=-(3-2x)$ and the obtaining of the wrong sign for x^2 should have been a notification to go back and find the error in the work, just as obtaining $+\sqrt{(x-4)}=-1$ should have been in the other equation.

I am sorry that this comment is so long, but I wished to touch on all the points brought up by Mr. Draughon; I should be glad to answer any further objections which he may care to make.

EDITORIALS.

The Electrical World, in its issue of October 13th, published an appreciative sketch with portrait of our subscriber, Alexander Macfarlane, LL.D., late of the Chair of Physics in the University of Texas, and now of Cornell University, Ithaca, New York.

Professor William C. L. Gorton, Ph.D., of the Chair of Mathematics and Astronomy, Woman's College of Baltimore, died on November 9th, aged 28 years. Professor Gorton was a member of the American Mathematical Society.

Professor E. W. Nichols, of The Virginia Military Institute, Lexington, Va., and author of a recent and valuable text-book on *Analytic Geometry*, writes as follows: "Permit me to acknowledge the benefit I have received from the MONTHLY in my capacity as a professor of Mathematics. I place it each month in the hands of my classes and find it a great stimulus to independent endeavor."

We acknowledge the receipt of a pamphlet of 17 pages, by our contributor, Prof. Warren Holden, Girard College. Philadelphia, entitled "Oneness of Arithmetic," in which it is shown that most of the Rules of

Arithmetic are traceable to one principle, namely, Ratio and Proportion. The method of the pamphlet is especially useful in elucidating the rules involving Percentage. A copy of the pamphlet will be mailed free to any reader of the MONTHLY, who sends his address to the author as above.

We wish to close up all subscriptions for 1894 by January, 1st and we trust all who have as yet failed to remit will respond by sending in the price of subscription without further delay. We have about $200 out in subscriptions for 1894, and we will need this money to meet the expenses of the MONTHLY which will have to be promptly paid at close of year. As you will be interested in making the MONTHLY a grand success for 189 , please lend us practical aid in closing up our books for the present year, so that it may not be necessary to carry over any liabilities into the next year.

Dr. George Bruce Halsted closed his inaugural address as President of the Texas Academy of Science in the following words: "From all this what do we learn as to the characteristic quality of the highest teaching, as to the true function of a university? Unfortunately there are still some people so dull, so envious, so unscientific, so stupid, as to maintain that the highest aim of a university should be the training of young men and young women, where they use the word "training" in its repressive, inhibitive sense

The most profound discoveries of modern science unite in replacing this old "training" idea of education by one immeasurably higher, finer, nobler.

We now know that the paramount aim of teaching at every stage, and preeminently of the final stage, at the university, should be to help the developing mind, the developing character, the developing personality. Judicious, delicate, sympathetic help is now the watchword.

Even horses and dogs worth owning are no longer "broken"; they are "gentled." What has brought about this glorious change? Science the greatest achievement of human life, the one thing that puts today—the present —in advance of all past ages.

Not only by having subjugated the forces of nature to the dominion of mind, but also by its intellectual influence, science is fundamentally remodeling the life and thought of modern humanity.

Though science is the purest knowledge, yet even our estimate of knowledge has been changed by science. Mere acquirement is now considered an unworthy end or aim for endeavor. Action, production alone now receive our homage, now gives a life worth living; and, therefore each must aim either at the practical application of his knowledge or at the extension of the limits of science itself. For to extend the limits of science is really to work for the progress of humanity".

We would be pleased to publish the entire address should Dr. Halsted see fit to permit us to do so. It was only after urgent solicitation that he sent us the above. This has only whetted our appetites for the whole Address.

Books and Periodicals.

Journal de Mathematiques Elementaires. Publie par H. Vuibert. Paris: Libraerie Nony et Cie 17, rue des Ecoles.

The November number has 8 quarto double-column pages. and contains a

Mathematical paper, solutions to problems in Arithmetic, Algebra, Geometry, and Physics, closing with a list of nine problems proposed. We are pleased to have this excellent Journal on our exchange list. Price 6 fr. a year, J.M.C.

L'Intermediaire des Mathematiciens. Dirige par C. A. Laisant, et Emile Lemoine. Tome 1., Nos. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Monthly. Paris, Ganthier-Villars et Fils, quaides Grands—Augustins, 55.

In the October number Questions 302 to 343 are propounded and 20 pages of answers are published. This new Journal occupies a unique field and its list of contributors shows that it is filling a real want. Price 6 fr. a year. J.M.C.

The September number of Annals of Mathematics contains "A Problem in Mechanical Flight," by G. E. Curtis; "Note on the Expansion of a Function," by W. H. Echols: "On Linkages for tracing Conic Sections," by W. M. Strong; "Some Considerations on the Nine-point Conic and its Reciprocal," by Miss Fanny Gates: and, "Byerly's Fourier's Series and Spherical,Cylindrical, and Ellipsoidal Harmonics". by R. S. Woodward. Six problems are proposed. J.M.C.

The Educational Times, London, for November maintains the strength of its Mathematical Department. Fourteen solutions are published, and a fine list of 36 problems are proposed, six of which are reproduced from the MONTHLY. J.M.C.

El Progreso Matematico. Periodico de Matematicas Puras y Aplicadas. Director: Don Zoel G. de Galdeano, catedratico de la Uuiversidad de Zaragoza.

The October number of this excellent Monthly cotains several interesting papers, and many fine problems and solutions. J.M.C.

The Kansas University Quarterly for October contains 6 interesting papers, among which we especially mention "The Hessian, Jacobian, Steinerian, in Geometry of one Dimension," by H. B. Newson, and "A special Class of Connected Surfaces," by Arnold Emch. J.M.C.

The Mathematical Messenger. George H.Harvill, Editor, Tyler,Texas.

At the begining of the year the size of the MESSENGER was enlarged, and very much improved in form, but it is to be regretted that only two numbers have thus far been issued during the year. The last number contained several papers besides the usual list of problems and solutions. Price,$2.00. J.M.C.

The Monist. A Quarterly Magazine Edited by Dr. Paul Carus. $2. per year, The Open Court Publishing Co., Chicago, Illinois. J.M.C.

The October issue contains among other valuable papers. "On the Principle of the Conservation of Energy," by Prof. Ernst Mach: on the Nature of Motion," by Major J. W. Powell. Nearly every issue of this valuable publication contains one or more important mathematical papers. J.M.C.

The School Visitor, September-October, Jno. S. Royer, Editor, Versailles, Ohio, containes 6 pages of solutions, together with much other interesting information. We regret that the editor has decided to discontinue this instructive journal at the close of the year. It will be greatly missed. J.M.C.

Miscellaneous Notes and Queries. A Monthly Magazine of History, Folk-Lore, Mathematics, Mysticism, Art, Sciences, etc.

The December Number, which has already reached us, contains the usual amount of varied and interesting information gathered from many sources. J.M.C.

Arithmetec by Grades. Books I., II., III., IV., V., VI., VII., VIII. For inductive teaching, drilling and testing. Prepared under the direction of John T. Prince. Boston, U. S. A. Ginn & Company, Publishers, 1894.

Some of the distinctive features of this series of books are the careful

gradation of problems; frequent reviews; the large amount of oral work: the great number and variety of problems; practicalness of work in respect to the character of problems, and the solution of them; the introduction of statistics and facts of physics, astronomy, history, geography, etc.; the use of drill tables and other devices; and a Teachers Manual whereby there is a separation of teacher's and pupil's books, so that pupils may be properly taught without being given too great assistance in the text. The books of this series have many decided merits and are well adapted for the purposes for which they were written. The price is 25 cents each. J.M.C.

An Elementary Treatise on Theoretical Mechanics. By Alexander Ziwet, Assistant Professor of Mathematics in the University of Michigan. Part III: Kinetics. Large 8vo, cloth. 236 pp. Price, $2.25. New York: Macmillian & Co.

This is the third part of Dr. Ziwet's Treatise on Theoretical Mechanics. The following subjects are very ably and clearly treated: Chapter V., Kinetics of a Particle. -(1) Impulses, (2) Rectilinear Motion, (3) Free Curvilinear Motion, (4) Constrained Motion, (5) Lagrange's Form of the Equations of Motion: Chapter VI., Kinetics of a Rigid Body.—(1) General principles, (2) Moments of Inertia and Principal axes, (3) Rigid Body with a fixed Axis, (4) Rigid Body with a fixed Point, (5) Free Rigid Body: Chapter VII., Motion of a Variable System. -(1) Free System, (2) System Subject to Conditions.

On pages 79-83, is a clear discussion of "The Problem of two Bodies." There are numerous lists of well selected and interesting problems throughout the book. The last chapter contains a brief sketch of the theory of Lagrange's generalized coordinates and Hamilton's principle. Each subject is treated in a way that will encourage rather than disparage the student. B.F.F.

Integral Calculus for Beginners with an Introduction to the Study of Differential Equations. By Joseph Edwards, M. A., Formerly Fellow of Sydney Sussex College, Cambridge. 8vo, cloth, 308 pp. Price, $1.10. New York: Macmillan & Co.

This volume is intended to give a sound introduction to a study of the Integral Calculus suitable for a student beginning the subject. The ordinary processes of integration, the principal methods of Rectification, Quadrature and Cubature are fully treated. The treating of the solutions of Differential Equations is especially commendable, as the student is thus early made acquainted with a subject which is greatly needed in his subsequent mathematical study. The text is abundantly supplied with well selected problems taken from various sources. The arrangement and presentation of the subject matter together with the typograpical execution of the work is first class. Persons desiring a good elementary text on the Integral Calculus should examine this work. B.F.F.

The Mathematical Visitor. Edited and Published by Artemas Martin, A.M., Ph.D., LL.D., Washington, D. C. Price, 50 cents for a Single Number.

Advanced sheets of the Mathematical Visitor for Jan. 1895, are before us. This issue of the Visitor will contain some excellent solutions to difficult Average and Probability Problems by the late Professor E. B. Seitz and others. These solutions are illustrated with beautiful diagrams. The Visitor will be ready for distribution about the last of January 1895 or as near that time as possible. B.F.F.

The Mathematical Magazine. Edited and Published by Artemas Martin, A.M., Ph.D., LL.D., Washington, D. C. Issued at Irregular Intervals. Price, $1.00 in advance, for four numbers.

Advanced sheets of the Mathematical Magazine for Jan. 1895 contains a paper, "About Cube Numbers whose Sum is a Cube Number," by Dr. Martin. This paper was read at the Summer Meeting of the American Mathematical Society held at Brooklyn, N. Y., Aug. 1894. This Number of the Magazine will be ready for distribution about the middle of January 1895.